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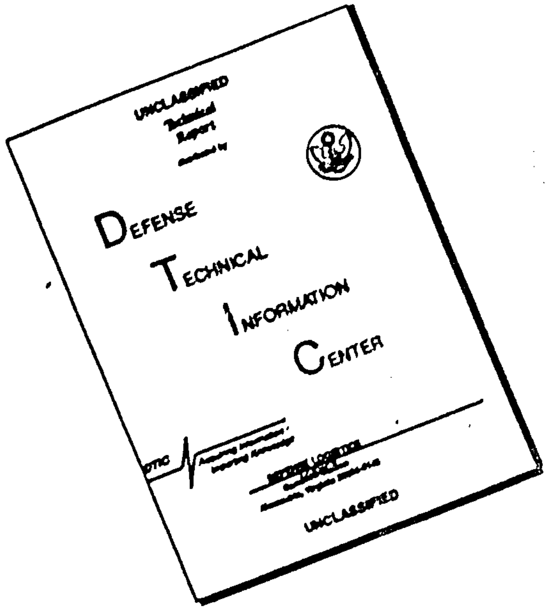

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(6) AN OPTIMAL INVENTORY POLICY FOR A

MILITARY ORGANIZATION

(10) by Edward B. Berman and
Andrew J. Clark

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AN OPTIMAL INVENTORY POLICY FOR A MILITARY ORGANIZATION

Edward B. Berman
Andrew J. Clark [1]

1. INTRODUCTION

The purpose of this paper is to present what is believed to be a realistic optimal inventory policy for a military organization. Our prime consideration has been practicability of application rather than elegance or generality. To further this end, we use the simplest mathematical techniques which provide the desired results and avoid unnecessary mathematical rigour. If we overelaborate in describing the concepts involved, it is only because we wish to engender more complete understanding and possible application of the theory in the "real world".

The supply system which we consider has two echelons of supply, which we shall call bases and depots, and are not unlike the wholesale and retail outlets of a large integrated company. Our prime assumption for the base stockage policy is that the concept of reorder level and fixed reorder amount is used. Thus, when the stock on hand at a base for a given item reaches the reorder level, R, an order is submitted to a depot for a fixed amount, Δ.[2] Our first problem will be to find the R and Δ which allows the base, as an isolated supply facility, to operate at least cost. We include both the case of recoverable items (items which can be reclaimed and repaired) and

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1/ While the authors are employed by The RAND Corporation, Santa Monica, the views expressed in this paper are their own.

2/ For conditions under which this policy is optimal, see On the Optimal Character of the (s,S) Policy in Inventory Theory, by A. Dvoretzky, J. Kiefer, and J. Wolfowitz in Econometrica, Vol. 21, No. 4, October, 1953, pp 586-596.

non-recoverable items. As a matter of general interest, we extend our results to include the case of a commercial firm using the reorder level concept. In our base results, we include all of the usual costs incurred by the base, and we also include two distinct kinds of pipeline time (the time between the issuance of a requisition and the receipt of the materiel). Our results also include the case of unknown demand, where the parameter of the demand density function is known only to within a probability function.

We then extend our results to include a system of military bases and depots. Here we discuss three general classes of procurement policies which are now in common use in military supply systems. These are life-of-type procurement, wherein sufficient spare parts for an end item are purchased at once to last the life of the end item; periodic procurement, where purchases are made at regular time intervals in sufficient quantity to satisfy anticipated needs during the period; and open-end contracting, where an order on a manufacturer is placed each time the system-wide stocks reach a certain minimum level. These procurement policies are characteristic of military organizations due to the allotment of funds on a fiscal year basis. For each procurement policy, we include the case of recoverable and non-recoverable items.

There are two additional system problems for which no solution is presented. These are the problem of finding the optimal ordering period for periodic procurement, and the problem of determining optimal delivery periods after procurement. This last problem is represented in the case where procurement is initiated at the first of the year for the entire year, and deliveries are made at specified times and in specified amounts throughout the year. This kind of procurement and delivery policy is in very common use.

As a method of implementing the proposed policies, one might think of

the existence of an information processing center in the supply system. This information processing center would maintain current balances of each item on hand at each of the bases and depots. As issues are made on the bases, the information processing center is so notified, and balances are adjusted at the end of the day for all issues during the day. The information processing center, having calculated the reorder level and amount of order for each item on each base, checks each balance after posting issues and determines whether or not the balance has gone below the reorder level. If so, the information processing center issues an order on a shipping activity to ship an amount to the base. The time between the issue at the base which caused the balance to fall below the reorder level and the acceptance of the materiel at the base is recognized as the routine pipeline time. If a base should exhibit a need for an amount not available at the base, (due perhaps to an unanticipated high demand) such need is relayed by the information processing center to a shipping activity. The time between the first recognition of the need at the base and the actual receipt of the materiel is the priority pipeline time. The information processing center, knowing the balances available and the consumption history of all the components of the system, is in a position to implement the policies which are subsequently derived. The policies are not contingent upon the existence of such an information processing center; however, the operation of the center would be consistent with the policies.

All of our results depend upon the existence of demand probability functions. These functions may be difficult to obtain in practice. They may depend upon past consumption, future programs, operational characteristics, and numerous other factors. On the other hand, any requirements and levels computational method requires such functions in one form or another. Such

functions are now being used whether realized or not. For example, the simple procedure of estimating the next sixty day's demand to equal the past sixty day's consumption automatically establishes a probability demand function, even though rather trivial in this case. Accepting, then, the fact that such functions must be established and used for any method, we address ourselves to the problem of using such functions to provide supply support in some "optimal" fashion. We assume these demand probability functions to be given; their actual derivation is outside the scope of this paper.

The general technique used in this paper establishes decisions based on cost considerations alone. We assume that any non-pecuniary elements involved in the decisions - principally the end item being out of commission for parts - can be converted into a cost factor. Behind this assumption lies a further assumption to the effect that the military organization can purchase more end items as an alternative to more logistics support, and vice versa. Thus, given a programmed requirement for the number of in-commission end items and for the activity of these end items, the assumed military objective is to minimize the sum of the costs of logistics support and the costs of a pool of end items being out of commission for parts. Alternatively, given a program of end item activity and a single fund limit for both the purchase of whole end items and for logistics support, the assumed military objective may be expressed as the maximization of the number of end items in commission. Using either statement of the military objective, a balance may be preserved between whole end items and logistics support by charging a depletion penalty against the logistics decisions. This depletion penalty consists of the total cost of the end item, including purchase and carrying costs, divided by the number of days in its expected useful life, and is assessed for each day the end item is out of commission for parts. This depletion charge, assessed

for each spare part which could cause the end item to be out of commission for lack of the part, is, in a sense, the contribution of the spare part to the pool of end items out of commission. Thus, no matter how many of the spare parts are stocked, there is some probability of these not being enough to keep the end item in commission, and therefore, each spare part has some expected contribution to the pool of end items out of commission for parts.

PART I - THE BASE

2. THE GENERAL SOLUTION FOR THE BASE[1]

In this section we consider the base as separate from the system. Implicit in the derivation of the base policy, are the following assumptions, most of which have been already suggested in the introduction:

A. The total issues which occur during the day are assumed to have occurred one at a time, and evenly spaced throughout the day.

B. The balance on hand is compared with the reorder level after each issue, and a requisition initiated if the balance on hand is equal to the reorder level. The requisition, each time, is for an amount, Δ , less any amounts received on priority requisitions during the previous process of replenishment. Also, the stock on hand when the order arrives is not less than the reorder level.

Assumption A and B taken together permit the further assumption that the balance on hand is exactly equal to the reorder level at the time the requisition is submitted.

C. The ability of depots to fulfill requisitions is assumed infinite. This does not imply that such is the case, but rather that any costs arising from depot failure should be assessed against the inventory policy of the system as a whole.

Assumption C allows us to look at the base as an isolated supply

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[1] The general approach taken in the base solution is similar to that of T. M. Whitin, Theory of Inventory Management, pp 56-62. We are indebted to T. M. Whitin for a critical review of a preliminary draft of this part of the paper. We are also indebted to R. Bellman, O. Morgenstern, and others for their comments on the preliminary draft of this paper. Perhaps we should also express our appreciation to K. J. Arrow, T. Harris, and J. Marschak, for it was their paper, Optimal Inventory Policy, Econometrica, Vol. 19, 1951, July, pp 250-272, which initially introduced this problem to us.

activity for our initial derivation. This assumption is modified when we consider the base as a component of the system.

D. Whenever demand exceeds the balance on hand, an emergency requisition is submitted for the excess of amounts demanded over amounts on hand. The emergency requisition requires premium communication and transportation.

E. The routine pipeline time, p, and the emergency pipeline time, ρ, are assumed to be constants.

Next, we will define the various costs encountered in the operation of the base, together with the symbols to be used in the derivation of the base policy. First, however, we should clarify our position in one respect. We derive our stockage policy on an individual item of supply basis; hence, our various costs are also on that basis. Since many of the costs are reduced, per unit, due to aggregation of items (such as the fixed paperwork cost of requisitioning, wherein if several items are included in the requisition, the per unit cost is less than the cost of a single item on the requisition), we assume that such reduced per unit costs are used. In practice, these reduced costs due to aggregation can be found by sampling techniques.

The symbols and component costs of supply used in subsequent calculations are:

(1) k = fixed costs of handling the paperwork and communications for a routine requisition.

(2) Δ = fixed amount of reorder.

(3) $p_1 + p_2\Delta$ = packaging, inspection, and handling costs, where p_1 is the fixed component and p_2 is the per unit cost associated with the size of the order, Δ.

(4) $t_1 + t_2\Delta$ = cost of transporting an amount Δ from the supply depot to the base.

(5) y = amount of stock on hand.

(6) $d_1 + d_2y$ = warehousing, depreciation, and obsolescence costs per unit of time for holding an amount y. These costs are also referred to as "holding" costs.

(7) σ = amount of depletion.

(8) p = routine pipeline time (time from when the balance on hand reaches the reorder level to the time of receipt of the materiel) = constant.

(9) ρ = priority pipeline time (time from the indication of need for an item unavailable at the base to the time of receipt of the item) = constant.

(10) $b_o + b_1q + b_2\rho q$ = depletion penalty, or the cost of understocking the item. In this cost, b_o is the fixed cost, if any, which is independent of the amount of depletion. The cost b_1 is the cost associated with the amount of depletion but not length of depletion. The component $b_2\rho q$ is the cost incurred through loss of utility of an end item being out of commission for ρ days. Here b_2 could be the cost per day of having an extra end item available for use. The component costs of the depletion penalty will be discussed in further detail later.

(11) x = number of the item demanded per unit of time. In subsequent work, x is a random variable.

(12) $f(x)$ = demand probability (density) function of the random variable, x. This function may be either continuous or be defined for only integral values of x.

(13) $F(x) = \int_0^x f(t)dt$ = the **integral** or cumulative demand probability function. Since $f(x)$ may be either a continuous function or a density function, such as a Poisson distribution, $f(x)$ must be integrable in the

Stieltjes sense over all intervals on the positive x-axis with zero as a lower limit. We also have the restriction that $F(x) \rightarrow 1$ as $x \rightarrow \infty$.

(14) R = reorder level. This is the point in the stock inventory where a routine requisition is required.

(15) θ = time period between the arrival of two successive orders.

We notice that most of the costs encountered by the base (in particular, items (3), (4), (6), and (10) above) are expressed as linear, non-homogeneous functions of the form $y = a + bx$. These costs, in reality, do not assume this form, but may be reasonably approximated by such functions. Actually, in our later formulation, we could just as easily consider these costs to be expressed as arbitrary functions expanded in power series. However, in practice, probably the best we can do is to find the linear approximations to the cost functions; hence, we restrict ourselves to such linear cost functions in our formulation.

In order to obtain a clearer understanding of the situation and to facilitate the calculation problem, we shall take the routine pipeline time as the fundamental time unit. This practice, in fact, is one of the main characteristics of our approach and provides a significant simplification of the problem. This unit, then, will also serve in the definition of those parameters defined in terms of units of time, such as the parameters d_1 and d_2, the variable x, and the requisitioning period, θ. Thus, for example, we speak of θ as being "so many routine pipeline times".

Our base inventory as a function of time may now be portrayed graphically as shown in Figure (1). Figure (1) represents a case of reality, where issues can be made several at a time and at any time during the day. By applying

assumption A, we amend Figure (1) to appear as shown in Figure (2). Thus, the issue of three items, causing a drop in the inventory from A to B in Figure (1), is represented in Figure (2) as three separate issues of one item each and at equal intervals throughout the day. Also, note that in Figure (2), the routine pipeline time, p, is considered to be an integral number of days; this is not at all necessary, but in practice such would probably be the case.

We are now in a position to write a function representing the cost of operation at the base for a given item and for a typical requisitioning period, θ:

$$(1) \quad L(\theta,R,\Delta) = (d_1 + d_2y)\,\theta + k + (p_1 + p_2\Delta) + (t_1 + t_2\Delta) + b_o\left[1 - F(R)\right]$$
$$+ (b_1 + b_2\rho)\int_R^{\infty} (x - R)\, dF(x).$$

The first term of the right member represents the cost of holding a quantity y in stock over a period of time θ. The next three terms are the handling, packaging, and transportation costs for the routine requisition. The term $b_o\left[1 - F(R)\right]$ represents the expected value of those costs of depletion which do not depend on the length of depletion nor amount of depletion. In this term, b_o is the fixed cost of depletion and $\left[1 - F(R)\right]$ is the probability of incurring the depletion, since $F(R)$ represents the probability of issuing R items or less during the pipeline period when routine replenishment is occurring, and $\left[1 - F(R)\right]$ represents the probability of demands being greater than R during that period. The term $(b_1 + b_2\rho)\int_R^{\infty} (x - R)\, dF(x)$ represents the expected costs of depletion which vary either with the quantity alone or with the quantity and duration of depletion. In this term, b_1 represents the costs which vary only with the quantity of depletion; $b_2\rho$ represents the costs which vary with both the quantity and the duration of depletion multiplied by the length of

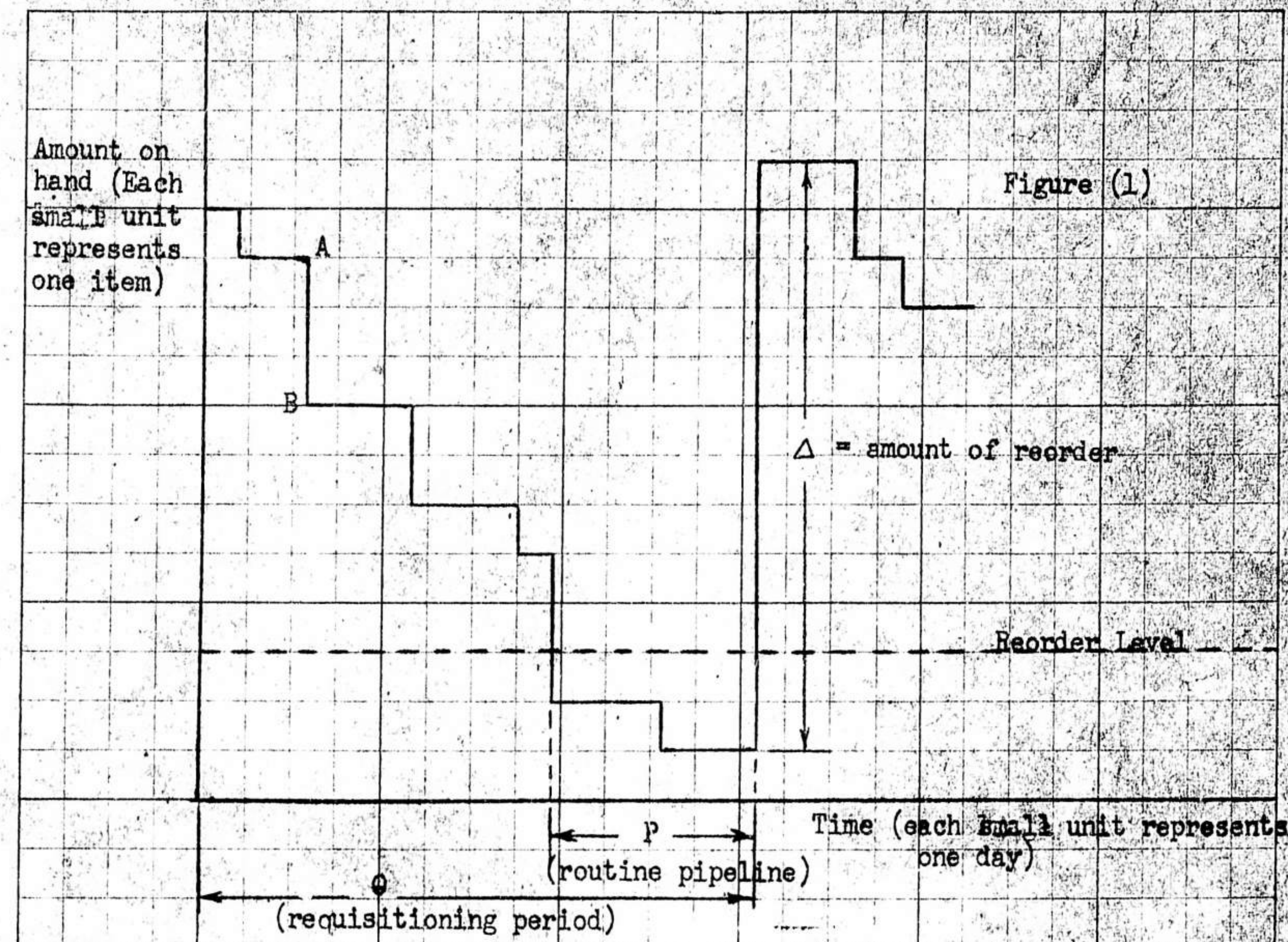


Figure (1)

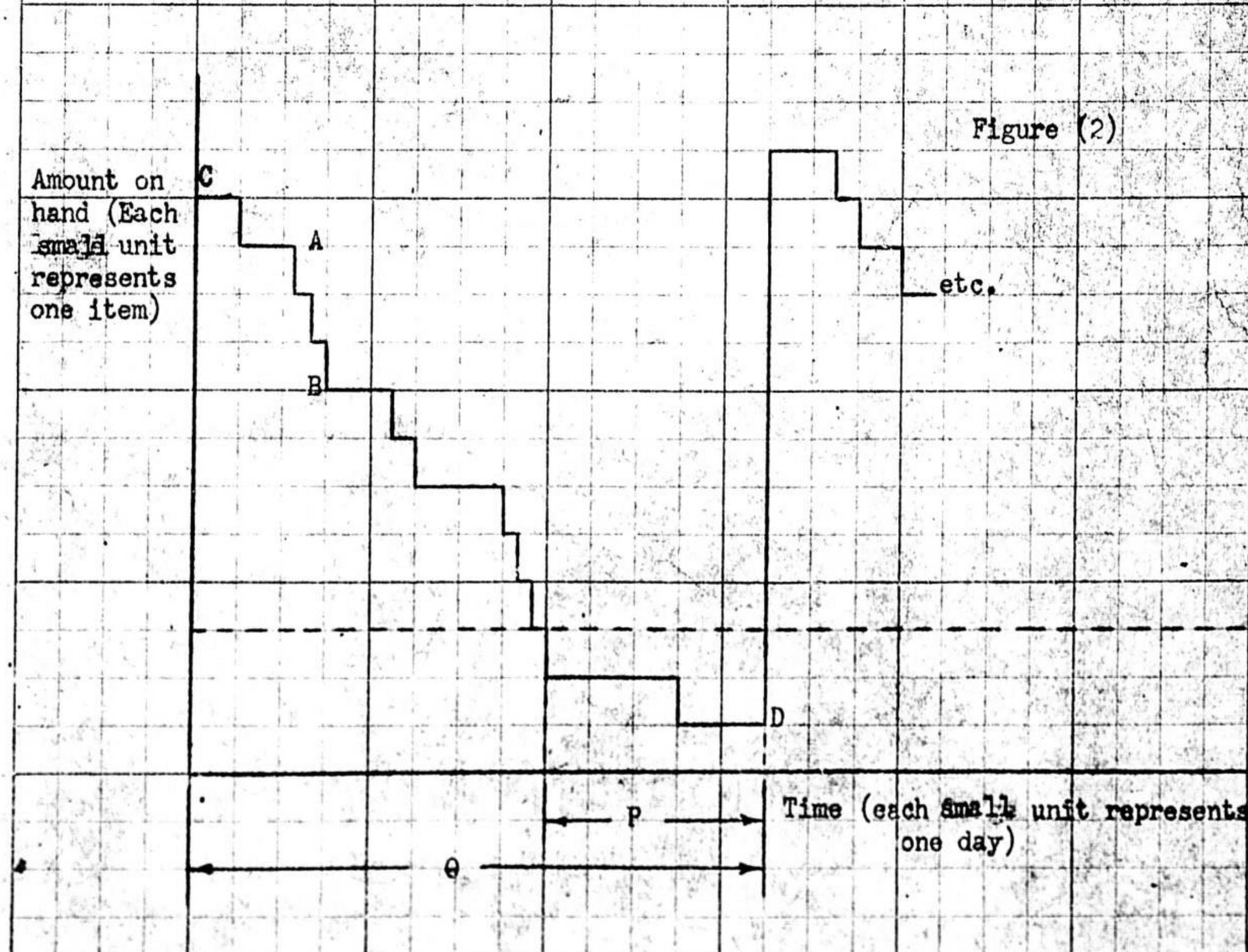
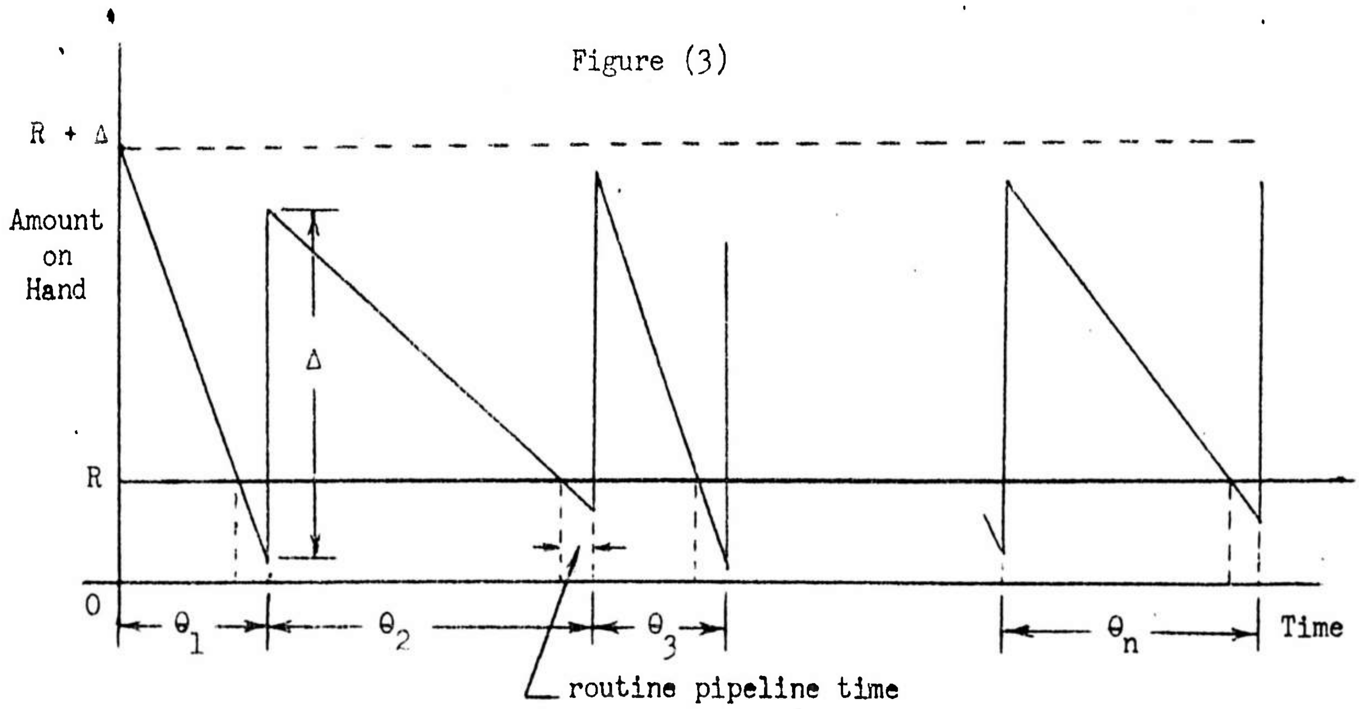


Figure (2)

depletion; $(x - R)$ is the quantity of depletion for a demand x; $dF(x)$ is the probability of the demand x ; and the **integral** from R to ∞ represents the summation of the products of all possible depletion sizes and their probabilities. Note that the two terms representing depletion costs are valid only under assumption B, in which it is assumed that the requisition is initiated when the balance on hand is exactly equal to the reorder level, R.

Now we shall consider a number of consecutive requisitioning periods, Θ_i (i = 1 to n). But first, in Figure (2), let us join with a straight line the stock level C at the beginning of the requisitioning period and the stock level D at the end of the requisitioning period. This line might be considered as representing a kind of "average" demand during the period Θ. If we construct such lines for the consecutive requisitioning periods Θ_i (i = 1 to n), we obtain the following picture:

Figure (3)



From Figure (3), we can get a clearer picture of our problem. If R is high, then we incur more holding costs. If R is low, the possibility of incurring the depletion cost is increased. If Δ is large, our holding costs

are again increased. If Δ is low, we requisition more often and increase the costs of requisitioning. Our problem, then, is to find the value of R and Δ which minimize the expected cost of supply per unit of time.1/ To find the expected cost of supply per unit time we must average the cost of supply over many requisitioning periods to allow for the variance of one requisitioning period from another. For this reason, we cannot merely minimize the cost expressed by equation (1), since this is only the cost of one requisitioning period.

Let T represent a period of operation of the supply activity, so that $T = \theta_1 + \theta_2 + \ldots + \theta_n = \sum_{i=1}^{n} \theta_i$, where θ_i is the time period between the i-th and the i+1-th receipt of materiel. Then the total cost over T is given by $\sum_{i=1}^{n} L(\theta_i, R, \Delta)$, where $L(\theta_i, R, \Delta)$ is the cost over the time period θ_i as given by equation (1). The cost per unit of time is then:

$$\text{cost per unit time} = \lambda = \frac{\sum_{i=1}^{n} L(\theta_i, R, \Delta)}{\sum_{i=1}^{n} \theta_i} = \frac{\sum_{i=1}^{n} \frac{L(\theta_i, R, \Delta)}{n}}{\bar{\theta}}$$

where we have divided numerator and denominator by n and let $\bar{\theta} = \frac{\sum_{i=1}^{n} \theta_i}{n}$, which is the average requisition period. Substituting from equation (1), we now obtain:

$$(2) \quad \lambda = (d_1 + d_2 y) + \frac{1}{\bar{\theta}} \left\{ k + (p_1 + p_2 \Delta) + (t_1 + t_2 \Delta) + b_o \left[1 - F(R) \right] \right\}$$

1/ We might remark here that we are interested in minimizing the averaged expected cost per unit of time rather than per requisitioning period. It can be shown that the value of Δ which yields least cost per requisitioning period is zero. This implies the average requisitioning period is also zero which in turn implies all requisitioning periods are zero. Thus, we do not obtain a useful solution to our problem.

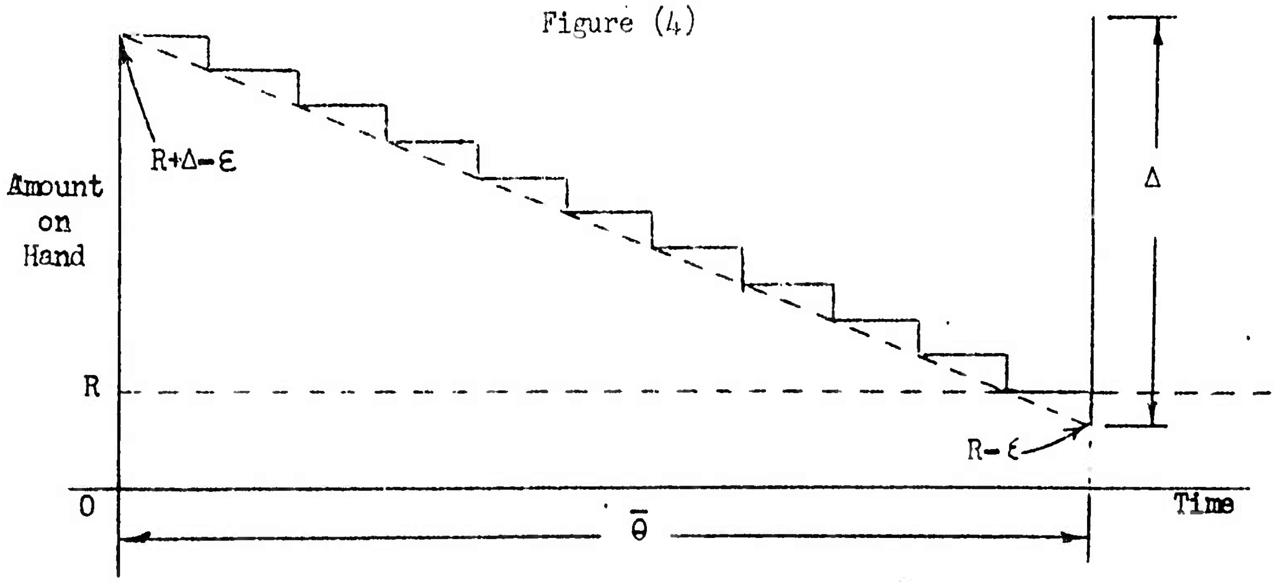
$$+ \frac{(b_1 + b_2\rho)}{\bar{\theta}} \int_R^{\infty} (x - R)\, dF(x),$$

where we have assumed y and Δ to be constant relative to the summation.

In equation (2), y becomes the average amount of stock on hand per unit time, obtained by including experience throughout T. The next thing we must do is express y in terms of R, Δ, and known quantities. To do this, we introduce the notion of an average requisitioning period. The length of this period is just $\bar{\theta}$, of course. The stock on hand at the beginning of this average period is $R + \Delta - \bar{x}$, where $\bar{x}$ is the average of the demands during the n routine pipeline time intervals, which immediately precede the receipt of the materiel. Similarly, the stock on hand at the end of the average period is $R - \bar{x}$.

Now let the number of requisitioning periods become infinite. Then $\bar{x} \rightarrow \epsilon = \int_0^{\infty} x\, dF(x)$, which is the average expected demand per routine pipeline time. Also, the balance on hand during the average requisitioning period will appear as follows:

Figure (4)



Notice that the step-function character of Figure (4) is due to the fact that

we issue one item at a time, and cannot issue fractions of items. If the amount on hand were a linear function, as represented by the diagonal dotted line in Figure (4), the average amount on hand would just be $R - \epsilon + \frac{\Delta}{2}$. However, we must add a correction term of $\frac{1}{2}$ to allow for the step-function effect. Thus, we obtain

(3) $y = R - \epsilon + \frac{\Delta+1}{2}$. [1/]

The value for y in equation (3) validates our prior assumption that y be constant relative to the summation in equation (2). It might also be noted that we are charging the warehousing costs for only an amount y whereas warehouse space is needed for the maximum amount stocked. We feel our assumption justified on the basis that all items in a warehouse will not be stocked in their maximum amounts at the same time, but will indeed average out as assumed, with some items requiring space for more than the amount assumed and others less in any given period of time.

Next we will establish a relationship between Δ and $\bar{\Theta}$. If we divide T into n equal intervals, each of length $\bar{\Theta}$, we can write

$$\bar{x}_i = \frac{\sum_{k=1}^{\bar{\Theta}} x_{ik}}{\bar{\Theta}}$$

as the arithmetic mean demand for the i-th period. Therefore,

$\bar{\Theta}\,\bar{x}_i = \sum_{k=1}^{\bar{\Theta}} x_{ik}$ and summing over i, we get

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1/ A more rigorous development of equation (3) is as follows: Define the requisition period, Θ_i, to be from the i-th time the balance reaches the reorder level to the i+1-th time. Then the average stock on hand during the k-th day of Θ_i is given by

$$y_{ik} = R - \sum_{j=1}^{k} x_{ij} + \frac{1}{2}x_{ik} + \frac{1}{2} \qquad (1 \leq k \leq p)$$

(3a) $$y_{ik} = R + \Delta - \sum_{j=1}^{k} x_{ij} + \frac{1}{2}x_{ik} + \frac{1}{2}, \qquad (p < k < p\,\Theta_i)$$

where we are temporarily letting x represent a day's demand rather than the

$$\bar{\Theta} \sum_{i=1}^{n} \bar{x}_i = \sum_{i=1}^{n} \sum_{k=1}^{\bar{\Theta}} x_{ik} = \sum_{i=1}^{n} \sum_{j=1}^{\theta_i} x_{ij} = n\,\Delta,$$

which is the total demand during T. Employing the extreme left and right members,

$$(4) \quad \Delta = \bar{\Theta}\, \frac{\sum_{i=1}^{n} \bar{x}_i}{n} = \bar{\Theta}\, \bar{x}.$$

Estimating $\bar{x}$ by ϵ, we substitute the values for y and $\bar{\Theta}$ from equations (3) and (4), respectively, into equation (2) and get

$$\lambda(\Delta,R) = d_1 + d_2\left(R - \epsilon + \frac{\Delta + 1}{2}\right) + \frac{\epsilon(k + p_1 + t_1 + b_0)}{\Delta} + \epsilon(p_2 + t_2)$$

$$- \frac{\epsilon\, b_0}{\Delta} F(R) + \frac{\epsilon(b_1 + b_2\rho)}{\Delta} \int_R^{\infty} (x - R)\, dF(x).$$

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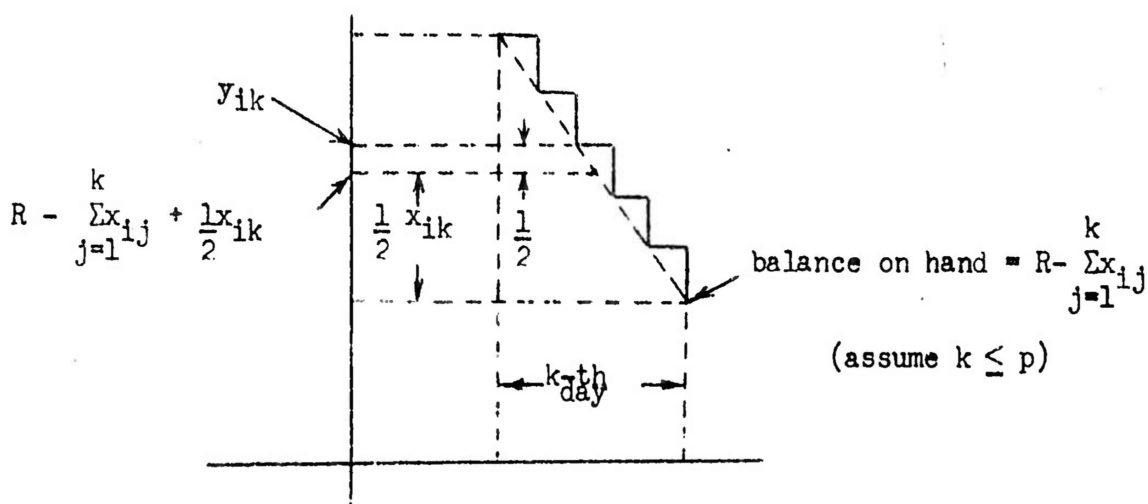
demand per pipeline time as previously defined. To show the derivation of these equations more clearly, we might isolate the k-th day, which would appear as follows:



Notice, in particular, the correction factor 1/2 in the diagram. In general, if we issue m items per issue and n issues per day, equally spaced throughout the day, then the amount on hand during the time period between the i-th and the i+1-th issues is just mi ($i = n, n - 1, \ldots, 1$) plus the amount on hand at the end of the period. The average amount on hand during the day is then

This equation can be rearranged to get

$$(5)\quad \lambda(\Delta,R) = A + \frac{\epsilon(B + b_o)}{\Delta} + \frac{d_2}{2}\Delta + d_2R - \frac{\epsilon b_0}{\Delta} F(R) + \frac{\epsilon C}{\Delta}\int_R^{\infty} (x - R)\, dF(x),$$

where

$$A = d_1 + \epsilon(p_2 + t_2) + d_2\left(\frac{1}{2} - \epsilon\right)$$

$$B = k + p_1 + t_1$$

$$C = b_1 + b_2\rho.$$

The function λ is minimized by equating to zero the partial derivatives of λ with respect to Δ and R and then solving these equations simultaneously to provide optimal values for Δ and R. Therefore, we obtain

$$(6)\quad \frac{\partial\lambda}{\partial R} = d_2 - \frac{\epsilon b_0}{\Delta} f(R) + \frac{\epsilon C}{\Delta}\frac{\partial}{\partial R}\int_R^{\infty} (x - R)\, dF(x) = 0$$

$\sum_{i=1}^{n} \frac{mi}{n} = \frac{mn(n+1)}{2n} = \frac{mn}{2} + \frac{m}{2}$ (plus amount on hand at the end of the period). Hence, the correction factor is $\frac{m}{2}$. Under assumption A, of course, $m = 1$ and the correction factor is $\frac{1}{2}$. From the equations (3a), we get the average amount on hand during θ_i to be:

$$\bar{y}_i = R + \frac{1}{2} + \frac{(\theta_i - 1)}{\theta_i}\Delta - \sum_{j=1}^{p\theta_i} \frac{(p\theta_i - j + 1)\, x_{ij}}{p\theta_i} + \frac{1}{2}\sum_{k=1}^{p\theta_i} \frac{x_{ik}}{p\theta_i}$$

or

$$(3b)\quad \bar{y}_i = R + \frac{1}{2} - \frac{\Delta}{\theta_i} + \sum_{j=1}^{p\theta_i} \frac{(j - \frac{1}{2})\, x_{ij}}{p\theta_i},$$

where we have set $\Delta = \sum_{j=1}^{p\theta_i} x_{ij}$. However, by so expressing Δ, we infer that the balance on hand goes negative in response to demands after total stock depletion. In actuality, balances do not go negative; but the correction term to be added to the value for y which we obtain by allowing negative balances, can be shown to be:

$$(7)\quad \frac{\partial\lambda}{\partial\Delta} = \frac{d_2}{2} - \frac{\mathcal{E}(B + b_0) - \mathcal{E} b_0 F(R) + \mathcal{E} C \int_R^{\infty} (x - R)\, dF(x)}{\Delta^2} = 0$$

We may solve explicitly for Δ in equations (6) and (7) and get

$$(8)\quad \Delta = \frac{\mathcal{E}}{d_2} \left\{ b_0\, f(R) - C \frac{\partial}{\partial R} \int_R^{\infty} (x - R)\, dF(x) \right\}$$

$$(9)\quad \Delta^2 = \frac{2\mathcal{E}}{d_2} \left\{ B + b_0 \left[1 - F(R)\right] + C \int_R^{\infty} (x - R)\, dF(x) \right\}.$$

Now if Δ is eliminated from equations (8) and (9), the optimal reorder level, R^*, may be determined from the equation which results and substituted back in equation (8) or (9) to find Δ^*, the optimal amount of reorder. Whether or not the values so obtained yield the actual minimum supply cost depends upon an investigation of the second partial derivates. We will omit this investigation: for all reasonable demand functions, a valid minimum does exist.

$$\frac{\mathcal{E}}{\Delta} \int_R^{\infty} \frac{(x - R)(x - R - 1)}{4x}\, dF(x).$$

This term, upon inspection, is so small that it can be comfortably neglected. Returning to equation (3b), we take the weighted average over n requisitioning periods and obtain

$$(3c)\quad y = \frac{\sum_{i=1}^{n} \theta_i \bar{y}_i}{\sum_{i=1}^{n} \theta_i} = R + \frac{1}{2} - \frac{\Delta}{\bar{\theta}} + \sum_{i=1}^{n} \sum_{j=1}^{p\theta_i} \frac{(j - \frac{1}{2})\, x_{ij}}{np\bar{\theta}}$$

By re-subdividing our interval T into n equal intervals, each of length $\bar{\theta}$, we can rearrange terms in the last member of the above equation to get

$$\sum_{i=1}^{n} \sum_{j=1}^{p\theta_i} \frac{(j - \frac{1}{2})\, x_{ij}}{np\bar{\theta}} = \sum_{i=1}^{n} \sum_{k=1}^{p\bar{\theta}} \frac{(k - \frac{1}{2}) x_{ik}}{np\bar{\theta}} = \sum_{k=1}^{p\bar{\theta}} \frac{(k - \frac{1}{2}) \sum_{i=1}^{n} \frac{x_{ik}}{n}}{p\bar{\theta}}$$

$$= \sum_{k=1}^{p\bar{\theta}} \frac{(k - \frac{1}{2})\, \bar{x}_k}{p\bar{\theta}}$$

But $\bar{x}_k \rightarrow \frac{\mathcal{E}}{p}$ for each k as $n \rightarrow \infty$. Substituting, we get

The calculation of R^* and Δ^* from equations (8) and (9) is a relatively easy matter, particularly on the electronic calculators. In fact, for some simple demand functions, explicit answers for R^* and Δ^* can be obtained. Examples of such functions are discussed in a later section.

The case of recoverable-type items at the base can be included in the above results by just considering the demand function, $f(x)$, to represent the net loss due to items condemned and items beyond base repair. This is due to the fact that if an item is base-reparable, it can readily be converted to a serviceable item - more readily, in fact, than obtaining the item from any other source. Hence, a base-reparable item can be treated as serviceable from a stock policy point of view.

In the derivation of our base results as expressed by equations (8) and (9), we used the idea of the number of requisitioning periods becoming infinite. One might question whether the results are tenable based upon such processes, inasmuch as it is certain the supply system will not operate forever. Actually,

---

$$\sum_{i=1}^{n} \sum_{j=1}^{p\theta_i} \frac{(j - \frac{1}{2})x_{ij}}{np\bar{\theta}} = \sum_{k=1}^{p\bar{\theta}} \frac{(k - \frac{1}{2})\epsilon}{p^2\bar{\theta}} = \frac{[p\bar{\theta}\,(p\bar{\theta} + 1) - p\bar{\theta}]\epsilon}{2p^2\bar{\theta}}$$

$$= \frac{\epsilon\,\bar{\theta}}{2}$$

Substituting back into equation (3c), we get

(3d) $y = R + \frac{1}{2} - \frac{\Delta}{\bar{\theta}} + \frac{\epsilon\,\bar{\theta}}{2}$.

Subsequently, we show that $\Delta = \epsilon\,\bar{\theta}$ so that equation (3d) becomes

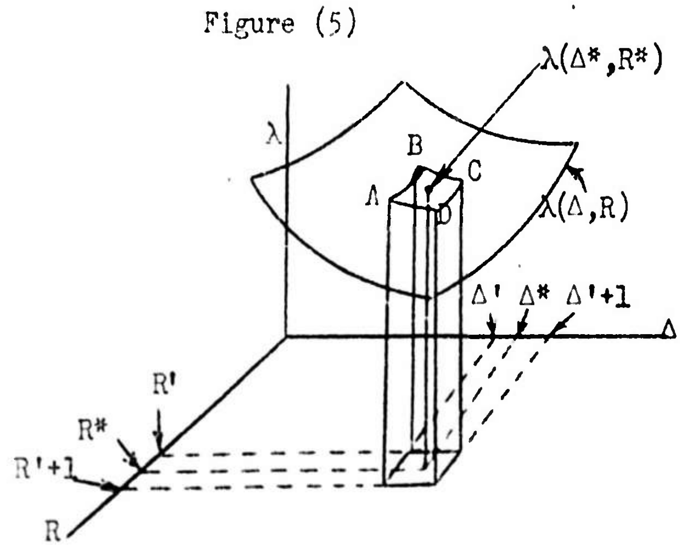
$$y = R - \epsilon + \frac{\Delta+1}{2} .$$

however, the method is validated by an implicit assumption, namely that our demand probability function is quite independent of time. Thus, even though reason might assess a zero probability of issuing n items a thousand years from now, this probability is not reflected in our demand function. Indeed, our demand function assesses the probability of demand to be the same for all time. But since our problem was to establish some Δ and R to use in the stockage policy, the use of Δ^* and R^* as calculated above is the best we can do if the given demand function, $f(x)$, is the best estimate of demand probability that we can obtain. If the demand probability function can be expressed in terms of time, then this becomes another problem for which we have no general solution. On the other hand, we can approximate a solution by recalculating R^* and Δ^* periodically with a demand function adjusted to reflect the trend established by past consumption and other factors. The derivation of the demand probability functions, however, is beyond the scope of this paper.

3. ROUNDING TO INTEGRAL VALUES

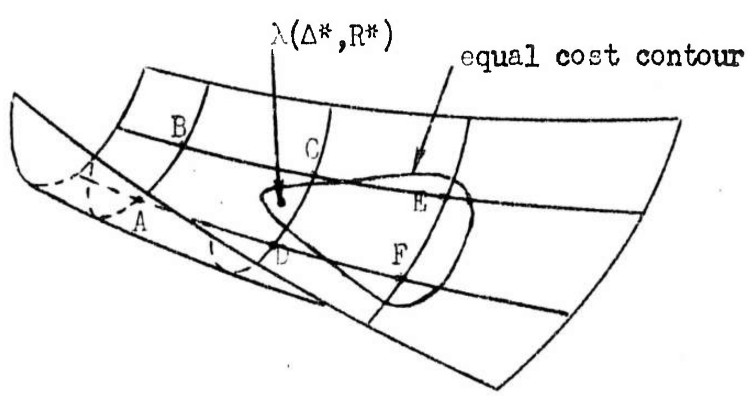
For practical application of the optimal reorder level, R^*, and reorder amount, Δ^*, we must examine the problem of rounding them to integral values. If we construct our cost surface, $\lambda(\Delta,R)$, in the vicinity of (Δ^*,R^*), we might obtain the situation shown in Figure (5).

Figure (5)



In Figure (5), R' and Δ' represent the largest integers contained in R^* and Δ^* respectively. The values, R', $R' + 1$, Δ', $\Delta' + 1$, determine the rectangle ABCD on the cost surface. At first glance, it might seem justified to round Δ^* and R^* to those integral values of Δ and R which yield the least cost among the four values, $\lambda(R', \Delta')$, $\lambda(R', \Delta'+1)$, $\lambda(R'+1, \Delta')$, and $\lambda(R'+1, \Delta'+1)$ which occur at the points A, B, C, and D. On the other hand, it is entirely conceivable that the cost surface, $\lambda(\Delta, R)$, in the vicinity of (Δ^*, R^*) might appear as follows:

Figure (6)



From Figure (6), it is clear that it is quite possible for the least cost λ, for integral values of Δ and R, to occur not at the corners of the rectangle ABCD containing $\lambda(\Delta^*, R^*)$, but at the points E or F or, in some cases, at even more remote points on the surface.

In general, then to find the integral values for Δ and R which afford the least cost of supply, we must proceed as follows: First we find that value, Δ_1, which minimizes the value $\lambda(R = R', \Delta)$. We do this by substituting R' for R in equation (7), the partial derivative of λ with respect to Δ, and solve for Δ, the solution being Δ_1. We then repeat this process by substituting $R' + 1$ for R in equation (7) to find Δ_2, that value of Δ which

minimizes $\lambda(R = R'+1,\Delta)$. Continuing this procedure, we substitute Δ' for Δ in equation (8), the partial derivative of λ with respect to R, and solve for R_1, the value of R which minimizes $\lambda(R, \Delta = \Delta')$. Similarly, we find R_2, the value of R which minimizes $\lambda(R, \Delta = \Delta' + 1)$. We have now found the points (R', Δ_1), $(R' + 1, \Delta_2)$, (R_1, Δ'), and $(R_2, \Delta' + 1)$ on the λ - surface, which represent the minimum values for λ along the extended "sides" of the rectangle ABCD in Figure (5). If we let the symbol $[x]$ represent the largest integer contained in x, we can now set down the following eight values for λ :

(C_1) $\lambda(R', [\Delta_1])$

(C_2) $\lambda(R', [\Delta_1]+1)$

(C_3) $\lambda(R'+1, [\Delta_2])$

(C_4) $\lambda(R'+1, [\Delta_2]+1)$

(C_5) $\lambda([R_1], \Delta')$

(C_6) $\lambda([R_1]+1, \Delta')$

(C_7) $\lambda([R_2], \Delta'+1)$

(C_8) $\lambda([R_2]+1, \Delta'+1)$

Of the eight values of λ so obtained, the least one provides the integral values of Δ and R to be used. The conditions on the nature of the surface, $\lambda(\Delta,R)$, which validate this process are assumed to exist and do exist for any practical case.

It might be remarked that the situation depicted in Figure (6) can occur in practice only for low values of Δ^* and R^*. For large values of Δ^* and R^*, the cost surface becomes so flat that one can automatically round to the nearest integer. For a region of the surface between these extremes, only the four values, $\lambda(R',\Delta')$, $\lambda(R'+1,\Delta')$, $\lambda(R',\Delta'+1)$, and $\lambda(R'+1,\Delta'+1)$ need be computed and compared.

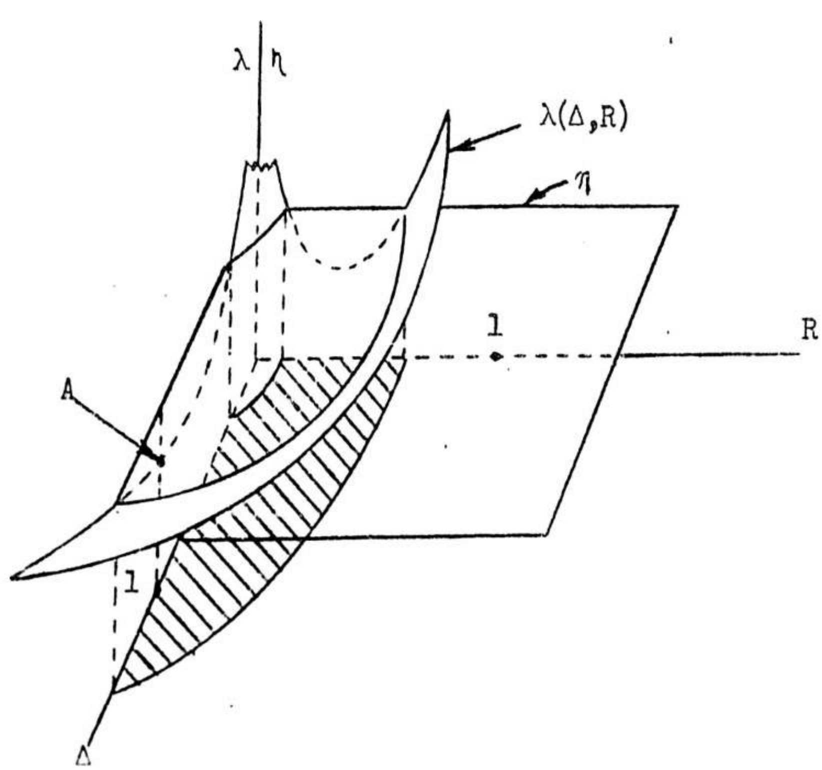
4. DETERMINING WHEN TO STOCK AT THE BASE

Since our cost function, $\lambda(\Delta,R)$ in equation (5), expresses the cost of holding an item of supply at the base, it is invalid for $\Delta = 0$, $R = 0$. This situation would be interpreted as not stocking the item at the base at all, but stocking the item at the depot instead. The average cost of supply per unit time, in this case, becomes

$$(10)\quad \eta = \epsilon(b_0 + b_1 + b_2\rho)$$

which is nothing more than the depletion penalty multiplied by the average probability of incurring it per unit of time. A superposition of the two cost surfaces represented by equations (5) and (10) might appear as follows:

Figure (7)



In Figure (7), the hatched area of the Δ-R plane represents those values of Δ and R for which the λ - surface lies below the η - plane. From

equation (5), we also notice the cost λ approaches infinity as Δ approaches zero, which renders the λ- surface invalid at (0,0). Of course this situation is reasonable; if Δ approaches zero so does $\bar{\Theta}$ and the fixed costs for each Θ_i occur more and more often until in the limit they occur infinitely often.

In Figure (7), it is clear that, with the scale values chosen, the cost A on the λ - surface (which is yielded by $\Delta = 1$, $R = 0$) would be chosen for any optimal Δ^* and R^* in the cross-hatched area. A very simple method, however, can be established to decide whether to stock at the base or not. This method is to compute the optimal, rounded Δ^* and R^* as outlined in sections 2 and 3, and then compare the resulting cost λ with the cost, h, of not stocking at the base. We notice, in this connection, that the cost h is easy to calculate, being nothing more than the cost of depletion multiplied by ϵ.

From our expression for λ in equation (5), it is fairly clear that the possibility of not stocking at the base can occur only for high-cost, low demand items or for low demand items with low depletion penalties. For low-cost, low demand items, the constant costs of requisitioning, in addition to any relatively high depletion costs, will cause the items to be stocked at the base. In fact, a high depletion cost for low-cost items will cause large stocks at the base even for very low demand rates.

5. OPTIMAL POLICY FOR UNKNOWN MEAN DEMAND

If, in the results of section 2, the form of the density function $f(x)$ is known but the mean demand ϵ can be expressed only as a probability function, $g(t)$, we can extend our theory as follows: First we calculate our optimal cost as a function of ϵ:

(11) $\lambda(\epsilon) = \lambda(\Delta^*, R^*, \epsilon)$.

Then, for each optimal Δ^* and R^* we multiply the cost $\lambda(\epsilon)$ by the prob-

ability of incurring that cost and sum over all such probabilities. In this manner we obtain

$$(12)\quad \gamma(\epsilon) = \int_0^\infty g(t)\ \lambda(\Delta^*(\epsilon), R^*(\epsilon), t)\ dt,$$

in which we changed the variable ϵ to t for the integration and remembered that Δ^* and R^* are themselves functions of ϵ. That value for ϵ, then, which gives least cost is calculated by the usual procedure of setting the first derivative equal to zero. Therefore, optimal Δ^* and R^* are given by

$$(13)\quad \begin{aligned} \Delta^* &= \Delta^*(\epsilon^*) \\ R^* &= R^*(\epsilon^*), \end{aligned}$$

where ϵ^* is a solution of the equation

$$\frac{d\gamma(\epsilon)}{d\epsilon} = 0 = \frac{d}{d\epsilon}\int_0^\infty g(t)\ \lambda(\Delta^*(\epsilon), R^*(\epsilon), t)\ dt.$$

The rounding procedure for Δ^* and R^*, in this case, becomes rather cumbersome. In theory, the procedure would proceed as follows: Let $R' = [R^*]$ (the largest integer contained in R^*), as previously defined. Proceeding as in section 3, we obtain $\Delta_1{}^*(\epsilon_1)$ as a solution of the equation $\frac{\partial\lambda(R', \Delta, \epsilon)}{\partial\Delta} = 0$. We then follow the procedure in the first part of this section to obtain

$$\gamma(\epsilon_1) = \int_0^\infty g(t)\ \lambda(R', \Delta_1^*(\epsilon_1), t)\ dt,$$

and then derive ϵ^*_1 as a solution of the equation $\frac{d\gamma(\epsilon_1)}{d\epsilon_1} = 0$. In this manner we get the costs

$(C_1)\ \lambda(R', \Delta_1)$ and $(C_2)\ \lambda(R', \Delta_2)$,

where $\Delta_1 = [\Delta_1^* \;(\epsilon_1^*)]$

$\Delta_2 = [\Delta_1^* \;(\epsilon_1^*)] + 1.$

By a similar process, we can derive the costs

$(C_3)\ \lambda(R'+1,\Delta_3)$ and $(C_4)\ \lambda(R'+1,\Delta_4)$,

where $\Delta_3 = [\Delta_2^*(\epsilon_2^*)]$

$\Delta_4 = [\Delta_2^*(\epsilon_2^*)] + 1$

$\Delta_2^* =$ the solution of $\dfrac{\partial\lambda(R'+1,\Delta,\epsilon)}{\partial\Delta} = 0.$

Through this method we can also get

$(C_5)\ \lambda(R_1,\Delta')$

$(C_6)\ \lambda(R_2,\Delta')$

$(C_7)\ \lambda(R_3,\Delta'+1)$

$(C_8)\ \lambda(R_4,\Delta'+1)$

wherein the roles of Δ and R have been reversed. The eight cost values so obtained can then be compared to find those integral values of Δ and R which yield least cost.

In practice, most of this procedure would be done by numerical methods, particularly if explicit values for R^* and Δ^* could not be obtained from equations (8) and (9). If this were the case, even R^* and Δ^* for known ϵ would probably be found by numerical methods.

One such numerical method, for ϵ given by $g(t)$, might be as follows: (a) Divide the interval $[a,b]$ on the t-axis of the function $g(t)$ into n equal sub intervals, where the points a and b might represent confidence limits at some level. Let ϵ_i $(i = 1$ to $n)$ represent the midpoints of the subintervals so obtained, with $g(\epsilon_i)$ denoting the associated probability. (Thus we approximate the continuous function, $g(t)$, by a density function,

where

$$g(\epsilon_i) = \int_{\delta_i} g(t)\,dt, \text{ and } \delta_i \text{ is the i-th subinterval).}$$

(b) For each ϵ_i, compute optimal rounded Δ and R as in section 2 and 3. Let these optimal values be denoted R_i^* and Δ_i^*.

(c) Compute the matrix $\lambda_{ij} = \lambda(\Delta_i^*, R_i^*, \epsilon_j)$. Form the matrix product

$$\sum_{j=1}^{n} \lambda_{ij}\, g(\epsilon_j) = \gamma_i. \quad (i = 1 \text{ to } n)$$

(d) Let γ_π represent the least of the values, γ_i. Then Δ_π^* and R_π^* provide the optimal values for Δ and R.

6. EXAMPLES

To illustrate previous results, we will now consider three examples of particular demand probability functions. In the first example, we are able to get explicit answers for both known and unknown mean demands. In the second, we can get explicit answers only for the case of known mean demand, while in the last example, we can go scarcely further than we have already.

Example 1. Let the demand probability function be of the form:

$$
\begin{aligned}
f(x) &= \alpha \quad x = 0 \\
&= \beta \quad x = 1 \qquad (14)\\
&= 0 \quad \text{all other } x.
\end{aligned}
$$

This demand function is very useful to express the case of very low demands providing we take β quite low, say less than 0.1.

Our condition that $\int_0^\infty f(x)dx = 1$ implies that $\alpha + \beta = 1$, or $\beta = 1 - \alpha$. Also we see that

$$\epsilon = \int_0^\infty x\, dF(x) = \beta$$

Since the last two terms of the cost function, equation (5), remain the same for all $R \geq 1$, it is clear that minimum cost can occur only for $R = 0$, or

$R = 1$, if only integral values for R are allowed. By direct substitution in equation (5), we get

$$\lambda(\Delta, R=0) = A + \frac{d_2}{2}\Delta + \frac{\beta(T + \beta\pi)}{\Delta}$$

(15)

$$\lambda(\Delta, R=1) = A + d_2\left(\frac{\Delta}{2} + 1\right) + \frac{\beta T}{\Delta}$$

where $T = k + p_1 + t_1$ and $\pi = b_o + C = b_o + b_1 + b_2\rho =$ total depletion cost. Equation (9) yields the values

$$\Delta^*_0 = \sqrt{\frac{2\beta(T + \beta\pi)}{d_2}} \qquad \text{(using } R = 0\text{)}$$

$$\Delta^*_1 = \sqrt{\frac{2\beta T}{d_2}} \qquad \text{(using } R = 1\text{)}$$

It is clear that the equation $\lambda(\Delta^*_0, R=0) = \lambda(\Delta^*_1, R=1)$ yields a β' such that for $\beta < \beta'$, $R = 0$ with Δ^*_0 provide least cost, and for $\beta > \beta'$, $R = 1$ with Δ^*_1 provide least cost. A calculation shows, in fact, that β' must satisfy the equation

$$[2\beta'^2\pi - d_2]^2 = 8\, d_2\beta' T.$$

For rounding, we compare the four values

$$\lambda([\Delta^*_0], R=0)$$

$$\lambda([\Delta^*_0]+1, R=0)$$

$$\lambda([\Delta^*_1], R=1)$$

$$\lambda([\Delta^*_1]+1, R=1)$$

and choose the integral Δ^* and R^* which provide the least value. The least value so obtained, $\lambda(\Delta^*, R^*)$ can then be compared against the cost $\eta = \beta(b_o + b_1 + b_2\rho)$ to decide whether or not to stock at the base. We would not

stock if $\eta < \lambda(\Delta^*, R^*)$.

Let us now consider the case where β, the arithmetic mean of $f(x)$, is given by a probability function, $g(t)$. Equation (12), together with equations (15) give

$$\gamma(\beta) = A + \frac{d_2}{2}\Delta_0^*(\beta) + \frac{\bar{g}T + \mu\pi}{\Delta_0^*(\beta)} \qquad (R = 0)$$

$$\gamma(\beta) = A + d_2\left(\frac{\Delta_1^*(\beta)}{2} + 1\right) + \frac{\bar{g}T}{\Delta_1^*(\beta)} \qquad (R = 1)$$

where

$$\bar{g} = \int_0^\infty t \; g(t)dt$$

$$\mu = \int_0^\infty t^2 \; g(t)dt.$$

The derivatives with respect to β yield

$$\left[\frac{d_2}{2} - \frac{\bar{g}T + \mu\pi}{\Delta_0^{*2}}\right]\frac{d\Delta_0^*}{d\beta} = 0 \qquad (R = 0)$$

$$\left[\frac{d_2}{2} - \frac{\bar{g}T}{\Delta_1^{*2}}\right]\frac{d\Delta_1^*}{d\beta} = 0 \qquad (R = 1)$$

from which, with equations (16), we find

$$\beta_0^* = \frac{-T + \sqrt{T^2 + 4\pi(\bar{g}T + \mu\pi)}}{\pi} \qquad (R = 0)$$

$$\beta_1^* = \bar{g}$$

wherein we have ignored negative values for β^*. We then calculate the four values,

(C_1) $\lambda(\Delta_0, R=0)$

(C_2) $\lambda(\Delta_0+1, R=1)$

(C_3) $\lambda(\Delta_1, R=1)$ where $\Delta_0 = [\Delta_0^*(\beta_0^*)]$

(C_4) $\lambda(\Delta_1+1, R=1)$ $\Delta_1 = [\Delta_1^*(\beta_1^*)]$

the least of which provides optimal integral values for Δ and R. The cost so obtained should be compared against the cost of not stocking at the base, which is given by $\eta = \bar{g}\,(b_0 + b_1 + b_2\rho) = \bar{g}\,\pi$

Example 2. Let the demand probability function be of the form

(18) $f(x) = \alpha e^{-\beta x}$.

This function is useful for somewhat higher demand rates than in the previous example, and is also useful in illustrating the procedure for the case of a continuous demand function.

Our condition that $\int_0^\infty f(x)dx = 1$ implies that $\alpha = \beta$. Also we calculate $\epsilon = \int_0^\infty x\, f(x)dx$ to be $\frac{1}{\alpha}$.

Substituting $f(x) = \alpha e^{-\alpha x}$ and $F(x) = \int_0^x f(t)dt = 1 - e^{-\alpha x}$ into equations (8) and (9), and eliminating Δ, we obtain

$$\left[e^{-\alpha R^*}\left(b_0 + \frac{C}{\alpha}\right)\right]^2 = \frac{2d_2}{\alpha}\left[e^{-\alpha R^*}\left(b_0 + \frac{C}{\alpha}\right) + B\right].$$

Let $z = e^{-\alpha R^*}\left(b_0 + \frac{C}{\alpha}\right)$. Then

$$z^2 - \frac{2d_2}{\alpha} z - \frac{2d_2}{\alpha} B = 0,$$

and

$$z = \frac{d_2}{\alpha}\left[1 \pm \sqrt{1 + \frac{2\alpha B}{d_2}}\right]$$

Since z is positive and all terms under the radical are positive, we take the plus sign for the radical and get

$$e^{\alpha R^*} = \frac{b_0\alpha + C}{d_2\left[1 + \sqrt{1 + \frac{2\alpha}{d_2} B}\right]}$$

having replaced our value for z. When substituted back into equation (8), we get

$$\Delta^* = \frac{e^{-\alpha R^*}(b_0\alpha + C)}{\alpha d_2} = \frac{1 + \sqrt{1 + \frac{2\alpha B}{d_2}}}{\alpha}.$$

Also, the optimal average requisitioning period is given by

$$\bar{\theta}^* = \frac{\Delta^*}{\epsilon} = \alpha\Delta^* = 1 + \sqrt{1 + \frac{2\alpha}{d_2} B}.$$

We can summarize our results by the equations

$$\bar{\theta}^* = 1 + \sqrt{1 + \frac{2\alpha}{d_2} B}$$

$$\Delta^* = \frac{\bar{\theta}^*}{\alpha}$$

$$R^* = \frac{1}{\alpha} \ln \frac{b_0\alpha + C}{d_2\bar{\theta}^*}$$

Integral values for Δ^* and R^* can now be found by the method of section 5. For the situation where the mean, $\frac{1}{\alpha}$, can be expressed only by a probability function, $g(t)$, an explicit answer for optimal Δ and R cannot be obtained; in practice, a numerical method would be used as suggested in section 5.

Example 3. Let the demand probability function be of the form

$$(20) \qquad f(x) = \frac{e^{-m} m^x}{x!} \qquad \text{(Poisson distribution).}$$

The Poisson function is useful to express most demands at bases, provided it is used on an item of supply basis as is done in this paper. This function, in particular, is much more realistic than either of the other two examples for higher demand rates.

We see that this function satisfies our condition that $\int_0^\infty f(x)\, dx = 1$ and we also calculate $\epsilon \int_0^\infty x\, f(x)\, dx$ to be just m. For the Poisson function, equations (8) and (9) become

$$(21) \qquad \Delta^* = \frac{m}{d_2} \left\{ b_0 \, f(R^*) + C \left[1 - F(R^*-1) \right] \right\}$$

$$(22) \qquad \Delta^{*2} = \frac{2m}{d_2} \left\{ B + \left[b_0 + C\,(m - R^*) \right] \left[1 - F(R^*) \right] + mC\, f(R^*) \right\}$$

Eliminating Δ, we get

$$(23) \qquad h(R^*) = m \left\{ b_0 f(R^*) + C \left[1 - F(R^*-1) \right] \right\}^2 - 2d_2 \left\{ B + \left[b_0 + C(m - R^*) \right] \left[1 - F(R^*) \right] + mC\, f(R^*) \right\} = 0.$$

This equation can be solved for R^* by numerical methods and R^* substituted back into equation (21) for Δ^*. The solution of equation (23) provides two integral values for R^*, one of which yields the least positive $h(R^*)$ and the other the greatest negative $h(R^*)$. These two values, R_1^* and R_2^*, when substituted into equation (21), yield Δ_1^* and Δ_2^*. These values for Δ are then rounded to $[\Delta_1^*]$, $[\Delta_1^*]+1$, $[\Delta_2^*]$, $[\Delta_2^*]+1$. Four values for λ are computed as follows:

(1) $\lambda([\Delta_1^*], R_1^*)$ (3) $\lambda([\Delta_2^*], R_2^*)$

(2) $\lambda([\Delta_1^*]+1, R_1^*)$ (4) $\lambda([\Delta_2^*]+1, R_2^*)$,

and those integral values for R and Δ which provide the least of the four costs are chosen for our optimal Δ and R.

For m expressed as a probability function, $g(t)$, the solution must be obtained by numerical methods as outlined in section 5.

7. MODIFICATION OF THE BASE COST EQUATION

The base cost equation, as expressed in equation (5), may be modified and refined in several respects. One such refinement has already been suggested in footnote 3.

Another correction to the equation should be made to allow for the possibility of a routine stock replenishment arriving while a priority requisition is being processed. Thus, if the priority pipeline time is 2 days, the cost equation as expressed charges a 2 day depletion penalty for a priority requisition submitted 1 day before the receipt of the routine replenishment. Of course, only 1 day's depletion penalty should actually be charged. The correction factor to allow for this overcharge is

$$\frac{b_2\rho}{2}\int_R^{\infty}(x - R)\, dF''(x),$$ where F'' is a demand probability function expressed in terms of $p - \rho$ as the unit of time. If $f(x)$ is a Poisson distribution expressed as

$$f(m,x) = \frac{e^{-m}m^x}{x!},$$

then $F''(x)$ is given by

$$F''(x) = \int_0^x f\left(\frac{p-\rho}{p}m, t\right) dt.$$

This correction factor, unlike that mentioned in footnote 3, may be of appreciable magnitude.

Another refinement in the base cost equation may be made by considering an addition assumption:

F. A priority request is not submitted for any depletion occurring in the last ρ days of the routine pipeline time.

Under this assumption, the base cost equation is further amended by substituting the function F'' for F in the b_o and b_1 terms of equation (5).

Another way in which our cost equation may be modified under further assumptions is in the manner of submitting priority recuisitions. Assumption A, which causes demand to occur one item at a time, together with assumption D, more or less implies that a priority requisition be submitted for each item demanded that is not available at the the base. In this case, our term b_o is zero (unless we wish to assess, in some way, the part of the capital costs necessary to establish facilities for the special handling of priority requisitions, whether or not such facilities are actually used in a particular case), and the term b_1 includes all costs of paperwork, handling, communications, packaging, inspection, and transportation for the priority request. Each item, of course, is packaged and shipped separately. However, it would seem more reasonable to institute another level, R_p, less than or equal to R, such that when this level is reached, a priority requisition be submitted for an amount Δ_p. The priority reorder level, R_p, may range from R down to some negative value. This concept, in fact, may be extended to provide as many reorder levels and reorder amounts as there are modes of transportation, and would in theory exactly specify the kind of transportation each shipment to the base would receive. However, to apply such a concept would require that the costs and pipeline time for each mode of transportation be constant; this requirement of course is unrealistic.

A more realistic approach toward the base stockage policy would be to

recognize two types of requisitions as represented by two general modes of communications and transportation. One type of requisition would be the routine requisition instituted when the amount on hand reaches the reorder level R. The pipeline time for this kind of requisition should be considered as a random variable to allow for the vagaries of slower communications and surface transportation. The other type of requisition would be the priority requisition instituted when the amount on hand reaches the reorder level, R_p. The pipeline time for this kind of requisition may be considered constant inasmuch as electrical communications and air transportation yield much less variable pipeline times. In this manner, we derive two possible modifications of our base cost equation:

(1) Inclusion of a priority reorder level, R_p, and a priority reorder amount, Δ_p.

(2) Expression of the routine pipeline time as a random variable instead of being constant.

The first modification is being subjected to further study. The second modification, however, can be readily accomodated by the procedure of section 5, providing the demand function is Poisson distributed. A characteristic of the Poisson distribution is that the probability of demand is proportional to the length of time. Therefore, we may express ϵ as $g(t)m$, where $g(t)$ is now interpreted as the probability function for the routine pipeline time, expressed in days, and m is the expected mean demand per day. This, of course, phrases the problem exactly as in section 5. Of course, we can still allow for the case of unknown mean demand in addition to random routine pipeline time by expressing m as $m(s)$, so that ϵ is given by $g(t)\,m(s)$. The procedure of section 5 still applies directly, but with double integration since the variables s and t are independent.

8. APPLICATION OF THE OPTIMAL INVENTORY POLICY TO A COMMERCIAL FIRM

The foregoing formulation of an optimal inventory policy is applicable to a profit-making firm with only minor modifications. We redefine our costs as follows:

(a) $d_3 + d_4 y$ = the cost of holding in inventory an amount y for one unit of time, including depreciation, obsolescence, warehousing costs, interest costs, and insurance.

(b) $p_1 + p_2\Delta + p_3\Delta^2$ = the cost of buying an order of size Δ.

(c) $b_0 + cq$ = the cost of depletion by amount q, including all present and future losses of custom, discounted to the current time period.

(d) M = the average gross utility of operating the inventory for one unit of time.

(e) All other symbols retain their former definitions.

We may now write the equation for the net loss, L, during a requisitioning cycle, θ, as:

$$L(R,\Delta,\theta) = (d_3 + d_4)\,\theta + k + (p_1 + p_2\Delta + p_3\Delta^2) + (t_1 + t_2\Delta) + b_0\left[1 - F(R)\right] + c\int_R^\infty (x - R)\,dF(x) - M\theta.$$

Since all costs and utilities in each requisitioning cycle, θ, are discounted to the current time period (c was the only parameter requiring time discounting) we may derive λ, the average net loss per unit time, as in section 2 and write:

$$\lambda(R,\Delta) = d_3 - M + d_4\left(R - \epsilon + \frac{\Delta + 1}{2}\right) + \frac{\epsilon(k + p_1 + t_1 + b_0)}{\Delta} + \epsilon(p_2 + t_2) + \epsilon p_3\Delta - \frac{\epsilon b_0}{\Delta}F(R) + \frac{\epsilon c}{\Delta}\int_R^\infty (x - R)\,dF(x),$$

having substituted $\frac{\Delta}{\epsilon}$ for $\bar{\theta}$. Thus, we obtain

$$(24)\quad \lambda(R,\Delta) = A' + \frac{\epsilon(B' + b_0)}{\Delta} + \frac{d'_2}{2}\Delta + d_4 R - \frac{\epsilon b_0}{\Delta} F(R) + \frac{\epsilon c}{\Delta}\int_R^\infty (x - R)\, dF(x),$$

where

$$A' = d_3 - M + \epsilon(p_2 + t_2) + d_4\left(\frac{1}{2} - \epsilon\right)$$

$$B' = k + p_1 + t_1$$

$$d'_2 = d_4 + 2\epsilon p_3.$$

Equation (23) is of the same form as equation (5), and the solution for optimal R and Δ follows as before.

PART II - THE SYSTEM

9. SYSTEM PROCUREMENT POLICIES

In this part of the paper, several concepts of optimization on a system-wide level will be introduced, where a system is defined as consisting of several bases and depots. These formulations apply to both non-recoverable and recoverable items, and will be set forth separately for life-of-type procurement, periodic procurement, and open-contract procurement.

Before we begin, let us further describe these three kinds of procurement policies. Life-of-type procurement consists of the purchase at one time of enough of a spare part for an end item, such as an airplane or a vehicle, to keep the end item in repair for its expected remaining life. Needless to say, the prediction of the expected life of an end item, let alone the prediction of what and how many spare parts will be needed, is an almost insurmountable endeavor; yet such a procurement policy is often warranted in view of the enormous retooling and setup costs to make parts for an end item no longer in production. This is particularly true with aircraft which have a notoriously short production period. Life-of-type procurement may also be used for very low cost items where the procurement costs (cost of contracting, etc.) are very much higher than the cost of the items.

Periodic procurement is the regular purchase of supplies to satisfy demands for them during a particular fixed period. Thus, we may procure materiel at the beginning of a year to last specifically through the year and supposedly no longer than the year. A variant of this policy occurs when several deliveries are specified during the period, perhaps at equal time intervals and for specified amounts at each delivery. Periodic procurement and its variants are very common in military organizations due

primarily to the budgeting and authorization of funds on a fiscal year basis.

Open-contract procurement, in its simplest form, consists of procuring "on demand", or when the stock balance reaches a certain warning level. Thus, it is very similar to the reorder point concept of our base policy. There are many variations in this kind of procurement, such as guaranteeing minimum amounts of orders, specifying different per unit prices for different size orders, etc. This type of procurement is very convenient for avoiding the high costs of contracting, but is more or less limited to items of common supply, or items where retooling and setup costs are relatively low. Also, in military organizations, there are budgetary and authorization restrictions on the use of this procurement policy.

A fourth type of procurement policy is possible; one which is not now in general use. This is the case where the procurement periods are variable rather than being fixed as in the case of periodic procurement. This policy uses a reorder level to trigger procurement, and is similar in concept to the base formulation. This type of procurement policy is discussed with the case of open-contract procurement, since the formulation is identical. It will be referred to as "variable period procurement".

The decision between the different kinds of procurement is a very difficult problem and is heavily dependent upon the particular production problems of the item. We do not attempt to solve this decision problem; we assume the type of procurement to have been somehow already established for each item. We are probably optimistic in saying that we defer this problem to future study.

10. THE SHADOW-PRICE CONCEPT

In the equation for average expected cost of supply at the base per unit of time (equation (5) in section 2), the coefficienct d_2 was introduced as that part of the warehousing, wear-and-tear depreciation, and obsolescence

costs which vary with y, the amount held at the base. This definition of d_2 needs to be reconsidered when the base is treated as part of a system of bases and depots.

First, let us assume that the item incurs the same warehousing, wear-and-tear depreciation, and obsolescence costs if it is stocked in a depot as if it were carried at the base. This seems reasonable, particularly in view of the fact that the obsolescence cost is incurred at the time of contracting for the item - the military system must face this obsolescence cost once it purchases the item, regardless of where the item is stocked. Let us agree, then, not to charge these costs to any of the bases but rather to a "system" cost, in which we will later include procurement costs, contracting costs, and other costs which pertain more to the system as a whole rather than to any component of the system.

On the other hand, if it does not cost the base anything to hold the items, then it is intuitively evident that an infinite base stockage offers least base cost, because the depletion cost becomes zero, and the paperwork, packaging, and transportation costs are incurred only once. This is also seen by inspection of equation (8), where if $d_2 \longrightarrow 0$, $\Delta \longrightarrow \infty$. Therefore, we must seek other factors which will restrict base stockages.

The most obvious restricting factor on base stockages is the fact that there is only a finite amount of the stock in the system as a whole. Let us suppose that the system is closed in the sense that stocks do not move in or out of the system after procurement has been made. Then it seems natural that the various bases should "compete" for the fixed amount of stocks available. So that they may compete equitably, we impose an internal "price" for the item and charge this price, for each unit of time, to the average amount stocked at the base whenever we use the results of Part I in calculating base levels.

After base levels are calculated, we do not include this factor in assessing base costs, since it is in fact not actually incurred. We call this internal price the "shadow" price because it is used only in effecting stockage distribution and has no particular relation to the actual procurement price for the item.

We now have generally identified two kinds of costs: costs associated with the bases and costs associated with the system as a whole. It is intuitively evident that the sum of these two kinds of costs is minimized by placing all the available amounts at the bases. Any extra amounts at the bases certainly reduce the base costs, while the costs attributed to the system remain the same (after procurement has been made), whether the items are kept at the depot or at the base. Therefore, we are in a position of incurring least over-all costs by having no depot stocks. This leads us to seek costs incurred by the system by not having depot stocks.

If we put all the stocks at the bases, then a particular base, depleting its amount on hand to its reorder level, can obtain a resupply only from other bases. If we suppose the depots to be the original source of the materiel when the initial distribution to the bases was made, then more costs are incurred by obtaining resupply from other bases than if the items were left at the depot to be used for resupply. In other words, for resupply to a base, it is cheaper to leave stocks at the depot for the resupply than to ship them to a base and then later ship them to another base requiring resupply. Any shipment from one base to another is an excess cost in this respect. We call the costs incurred by this trans-shipment of goods between bases, the "excess transportation costs", even though they include extra packaging and other associated costs.

Of course it may not be desirable to avoid all of these excess transportation costs. If we hold more stocks at the depots we cause more base

costs, mainly because we incur higher depletion costs at the bases. On the other hand, if we place few stocks in the depots, the increased base stockages provide lower base costs but higher excess transportation costs caused by trans-shipment between bases after depot stocks are depleted. There is some depot-to-base stockage ratio which should minimize expected cost to the system as a whole.

Let us next tie these concepts together and express them in symbolic forms. We begin by establishing some definitions and notations:

(1) Procurement Period = length of time between two successive arrivals of materiel from the factory after procurement. This definition is used particularly in context with the periodic procurement policy.

(2) Lead Time = length of time between the calculation of system requirements and the arrival of the materiel from the factory.

(3) Consumption Period = lead time plus the subsequent procurement period. This definition is used particularly in context with the periodic procurement policy.

(4) t = number of units of time in the procurement period.

(5) d = system shadow price, expressed in terms of the same unit of time as the procurement period.

(6) d_i = system shadow price, expressed in terms of the routine depot-to-base pipeline time for the i-th base.

= $p_i d$, where p_i is the routine pipeline time for the i-th base expressed in the procurement period unit of time.

(7) Y = expected total base stockages at the end of the consumption period.

(8) S_e = expected total system stockages at the end of the consumption period.

(9) S_o = total system stocks on hand at the time of requirements calculation, or at the beginning of the consumption period.

(10) P = amount procured.

(11) $S = P + S_o$ = amount which, if available at the time of requirements calculation, would satisfy system demands throughout the consumption period.

(12) J = excess transportation costs incurred during the procurement period, where excess transportation costs are defined as transportation, packaging, and associated costs of all materiel which is shipped from one base to another base. Transportation costs from depot to base are not included in J.

Now let us look at the base costs. First, we agreed to set d_2 to zero in the base cost equation, equation (5), in order not to charge the depreciation and other holding costs. We also do not include any factor involving the shadow price, because this is used only in calculating Δ and R. The actual direct costs incurred by the i-th base are then given by

$$(25)\quad \lambda_i'(\Delta_i, R_i) = A_i' + \frac{\varepsilon_i B_i}{\Delta_i} + \frac{\varepsilon_i b_{oi}}{\Delta_i}\left[1 - F_i(R_i)\right] + \frac{\varepsilon_i C_i}{\Delta_i}\int_{R_i}^{\infty}(x - R_i)\,dF_i(x),$$

where $A_i' = \varepsilon_i(p_{2i} + t_{2i}) + d_{1i}'$, and the other terms are as previous defined. Here, d_{1i}' represents only the fixed set-up costs encountered in including the item in base inventory, whereas in equation (5), d_1 also included the fixed part of the holding costs, assumed to be a linear function. In equation (25), Δ_i and R_i represent the optimal values calculated as in Part I from the following equations:

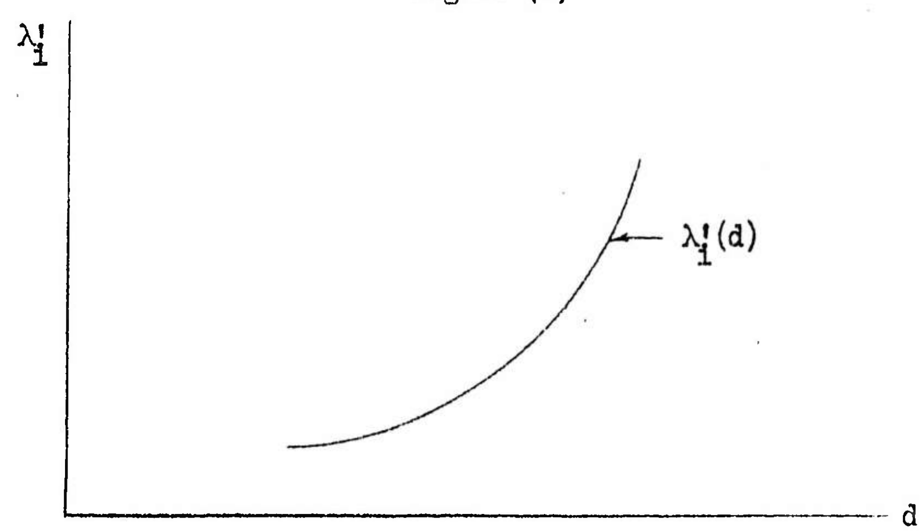
$$\Delta_i = \frac{\varepsilon_i}{d_i} \left\{ b_{oi} f_i(R_i) - C_i \frac{\partial}{\partial R_i} \int_{R_i}^{\infty} (x - R_i)\, dF_i(x) \right\}$$

(26)

$$\Delta_i^2 = \frac{2\varepsilon_i}{d_i} \left\{ B_i + b_{oi} \left[1 - F_i(R_i)\right] + C_i \int_{R_i}^{\infty} (x - R_i) dF_i(x) \right\}.$$

Therefore, both Δ_i and R_i become functions of d; hence λ_i' in equation (25) is a function of d. As we have previously noted, $\lambda_i'(d)$ increases as Δ_i and R_i decrease due to higher expected depletion costs. This implies that $\lambda_i'(d)$ increases monotonically as d increases. For the i-th base, $\lambda_i'(d)$ might appear as follows:

Figure (8)



The function may or may not be concave upward as shown.

In order to aggregate the values for λ_i' for n bases, we must first convert them to a comparable unit of time. It must be remembered that λ_i' ($i = 1$ to n) is the expected base stockage cost per unit of time, where the time unit is the routine depot-to-base pipeline time for that base. Therefore, we divide each λ_i' by p_i, the depot-base routine pipeline time

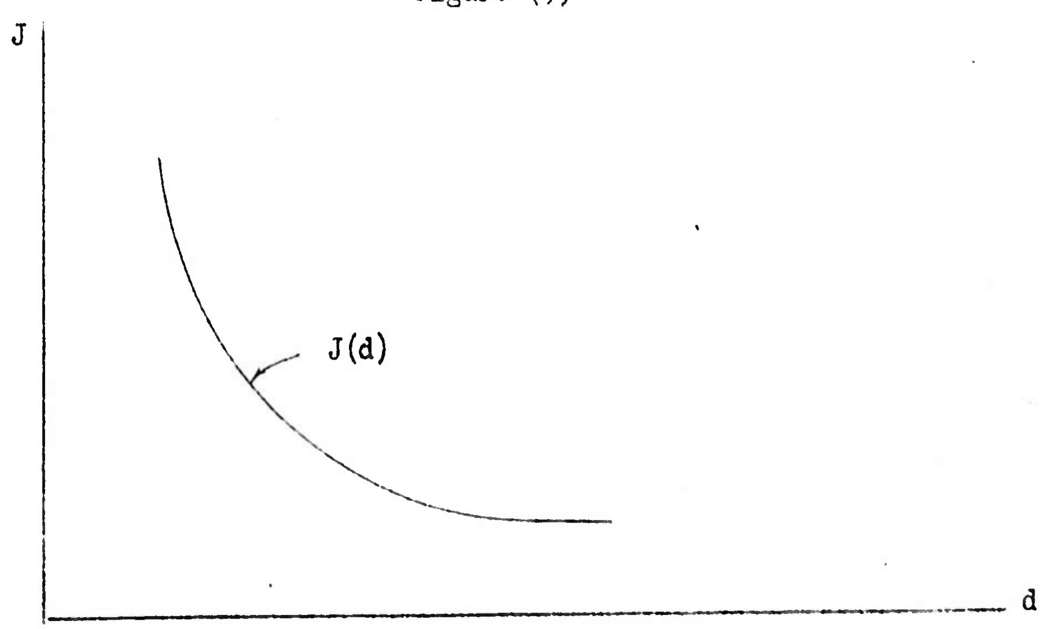
measured in terms of the same unit of time as the procurement period; multiply by t, the number of time units in the procurement period, and get our total expected base costs for the procurement period to be

$$(27)\quad \lambda_s = t \sum_{i=1}^{n} \frac{\lambda_i'}{p_i} \quad \text{for } n \text{ bases.}$$

Of course, λ_s is also a function of d since each λ_i' is, and λ_s will also increase monotonically as d increases, since this is true for each λ_i'.

The excess transportation costs J certainly decrease if the amount of base stocks decrease. This is due not only to the fact that fewer base stocks permit less trans-shipments between bases, but also to the fact that fewer base stocks provide more depot stocks which can be used to satisfy the base demands. Since we have previously observed that base stocks decrease monotonically as d increases, we conclude that J decreases monotonically as d increases. The excess transportation costs, expressed as a function of d, might then appear as follows:

Figure (9)



The function J may or may not be concave upward as shown.

The excess transportation costs also depend upon the amount procured, P. If we procure a large amount, then we increase depot stocks and decrease the costs J. If we next assume all other costs in our system, namely, the depot and procurement costs, to be functions of P only and given by the functions $D(P)$, then we can write:

$$(28) \qquad L'(P,d) = D(P) + J(P,d) + \lambda_s(d)$$

for the total cost, including procurement, of operating the system during the procurement period. Theoretically, values for P and d may be found which minimize this total system cost .[1]

The factor $D(P)$ in equation (28) is independent of base stockage distributions and depends entirely on the procurement costs and system depletion costs. Various functions for $D(P)$ that correspond to the different procurement policies will be discussed in sections 13 and 15.

11. THE GENERAL SYSTEM SOLUTION FOR NON-RECOVERABLE ITEMS

In seeking values for P and d which minimize the total system costs expressed by equation (28), we find that the function $J(P,d)$ for excess transportation costs is impossible to obtain. However, we can avoid its use by examining the circumstances which create these costs; namely, the ratio of base stocks to system stocks. Let us consider the ratio

$$(29) \qquad E = \frac{Y(d)}{S_e},$$

where $Y(d)$ is the expected base stockage at the end of the procurement period and S_e is the expected system stockage at the end of the procurement period. In equation (29) we consider E to be a fixed number less than or

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[1] Actually, there is an implicit restraint on optimal values for P and d; namely, that the total average expected base stocks, as determined by d, do not exceed the total available system stocks, as determined by P. This constraint is expressed by setting $E = 1$ in equation (29).

equal to 1. This equation should be considered as a means of "forcing" the amount procured to obtain a desired base-system stockage configuration at the end of the procurement period. The constant E may be set much less than 1 to provide depot stocks for war reserves or "insurance" or to conform with other command policies. Of course E may also be set equal to 1 to place all expected system stocks at the bases at the end of the procurement period.

We may now express the system costs as being

$$(30) \qquad L(P,d) = D(P) + \lambda_s(d),$$

subject to the restriction of equation (29). As will be subsequently demonstrated, equation (29) and (30) can be used to find both the amount of procurement, P and the subsequent stockage distributions for the various bases and depots. The remainder of this section will be devoted to obtaining equations from which the procurement amount P may be found.

In equation (29), we may express S_e as equal to $S - \ell$, or $P + S_o - \ell$ where ℓ is the expected consumption during the consumption period. The value S_o is the system stocks on hand at the time of requirements calculation, and hence known. The value ℓ is assumed to be a constant and is evaluated in section 13 for different procurement policies. Since equation (29) is now expressed in terms of d and P we may find the value of P which minimizes $L(P,d)$ in equation (30), subject to the restriction of equation (29), by the use of a Lagrangian multiplier. If μ is the Lagrangian multiplier, we form the function

$$\Gamma(P,d) = L(P,d) + \mu \left[Y(d) - E(P + S_o - \ell) \right]$$

and solve the equations

$$(31) \qquad \frac{\partial \Gamma}{\partial P} = \frac{\partial D(P)}{\partial P} - \mu E = 0$$

$$\frac{\partial \Gamma}{\partial d} = \frac{\partial \lambda_s(d)}{\partial d} + \mu \frac{\partial Y(d)}{\partial d} = 0$$

simultaneously with equation (29) for μ, P, and d. Eliminating μ in equations (31), we thus obtain the two equations

$$E \frac{\partial \lambda_s(d)}{\partial d} + \frac{\partial D(P)}{\partial P} \frac{\partial Y(d)}{\partial d} = 0$$

(32)

$$Y(d) - E(P + S_o - \ell) = 0$$

to be simultaneously solved for P.

The best estimate for $Y(d)$ to use in equation (31) is given by the sum over all bases of the average expected base stockages as given by equation (3). Substituting this sum in equations (32), and using equation (27) for λ_s we obtain the equations

$$\sum_{i=1}^{n} \left\{ \frac{tE}{p_i} \frac{\partial \lambda_i' \left[R_i(d), \Delta_i(d)\right]}{\partial d} + \frac{\partial D(P)}{\partial P} \left[\frac{\partial R_i(d)}{\partial d} + \frac{1}{2} \frac{\partial \Delta_i(d)}{\partial d} \right] \right\} = 0$$

(33)

$$\sum_{i=1}^{n} \left[R_i(d) - \mathcal{E}_i + \frac{\Delta_i(d) + 1}{2} \right] - E(P + S_o - \ell) = 0.$$

In the above equations, we must remember that $R_i(d)$ and $\Delta_i(d)$ are given by equations (26), and λ_i' is given by equation (25). The function $D(P)$ is derived in sections 13 and 15 for various procurement policies.

Equations (33) were obtained in the manner shown in order to retain the shadow-price d in a more explicit form. Actually, the shadow-price was used only as a heuristic device and can be readily eliminated in a mathematical solution. Such a method might be as follows: Eliminating d_i from

equations (26) we obtain:

$$(33a)\qquad \varphi_i(\Delta_i,R_i) = 2B_i + b_{oi}\left[2 - 2F_i(R_i) - \Delta_i f_i(R_i)\right]$$

$$+ C_i\left[2\int_{R_i}^{\infty}(x - R_i)dF_i(x) + \Delta_i \frac{\partial}{\partial R_i}\int_{R_i}^{\infty}(x - R_i)dF_i(x)\right] = 0$$

There are actually n such equations, one for each base, each with different values for optimal Δ and R.

We are now interested in solving the equation

$$L(P,\Delta_i,R_i) = D(P) + t\sum_{i=1}^{n}\frac{\lambda'_i(R_i,\Delta_i)}{p_i}$$

subject to the restrictions of the equations $\varphi_i(\Delta_i,R_i) = 0$ and the equation

$$\psi(\Delta_i,R_i,P) = \sum_{i=1}^{n}\left(R_i - \mathcal{E}_i + \frac{\Delta_i + 1}{2}\right) - E(P + S_o - \ell) = 0$$

We form the function

$$\Gamma(\Delta_i,R_i,P) = L(P,\Delta_i,R_i) + \mu_i\,\varphi_i(\Delta_i,R_i) + \mu\psi(\Delta_i,R_i,P)$$

where μ_i and μ are Lagrangian multipliers (i = 1 to n).

The partial derivatives of Γ set to zero are

$$\frac{\partial\Gamma}{\partial P} = \frac{\partial D(P)}{\partial P} - \mu E = 0$$

$$\frac{\partial\Gamma}{\partial R_i} = \frac{t}{p_i}\,\frac{\partial\lambda'_i(R_i,\Delta_i)}{\partial R_i} + \mu_i\,\frac{\partial\varphi_i(R_i,\Delta_i)}{\partial R_i} + \mu = 0$$

$$\frac{\partial\Gamma}{\partial\Delta_i} = \frac{t}{p_i}\,\frac{\partial\lambda'_i(R_i,\Delta_i)}{\partial\Delta_i} + \mu_i\,\frac{\partial\varphi_i(R_i,\Delta_i)}{\partial\Delta_i} + \frac{\mu}{2} = 0$$

Eliminating the Lagrangian multipliers, we get the equations

$$(33b) \qquad + \frac{Et}{p_i}\left[\frac{\partial\lambda_i'}{\partial\Delta_i}\frac{\partial\varphi_i}{\partial R_i} - \frac{\partial\varphi_i}{\partial\Delta_i}\frac{\partial\lambda_i'}{\partial R_i}\right] + \frac{\partial D(P)}{\partial P}\left[\frac{1}{2}\frac{\partial\varphi_i}{\partial R_i} - \frac{\partial\varphi_i}{\partial\Delta_i}\right] = 0,$$

which must be solved simultaneously with the equations $\varphi_i(\Delta_i, R_i) = 0$ and $\psi(\Delta_i, R_i, P) = 0$ to obtain values for P, Δ_i, and R_i.

Since $\dfrac{\partial\lambda_i'}{\partial\Delta_i} = -\dfrac{\varepsilon_i}{\Delta_i^2}\left[B_i + b_{oi}\left[1 - F_i(R_i)\right] + C_i\displaystyle\int_{R_i}^{\infty}(x - R_i)dF_i(x)\right]$

and $\dfrac{\partial\lambda_i'}{\partial R_i} = -\dfrac{\varepsilon_i}{\Delta_i}\left[b_{oi}f_i(R_i) - C_i\dfrac{\partial}{\partial R_i}\displaystyle\int_{R_i}^{\infty}(x - R_i)\,dF_i(x)\right]$,

we see that

$$\varphi_i(\Delta_i, R_i) = -\frac{2\Delta_i^2}{\varepsilon_i}\frac{\partial\lambda_i'}{\partial\Delta_i} + \frac{\Delta_i^2}{\varepsilon_i}\frac{\partial\lambda_i'}{\partial R_i} = 0$$

or

$$\frac{\partial\lambda_i'}{\partial R_i} - 2\frac{\partial\lambda_i'}{\partial\Delta_i} = 0.$$

Substituting $\dfrac{\partial\lambda_i'}{\partial\Delta_i} = \dfrac{1}{2}\dfrac{\partial\lambda_i'}{\partial R_i}$ in equation (33b), we get

$$\left[\frac{1}{2}\frac{\partial\varphi_i}{\partial R_i} - \frac{\partial\varphi_i}{\partial\Delta_i}\right]\left[\frac{Et}{p_i}\frac{\partial\lambda_i'}{\partial R_i} + \frac{\partial D(P)}{\partial P}\right] = 0.$$

It can be shown that

$$\frac{1}{2}\frac{\partial\varphi_i}{\partial R_i} - \frac{\partial\varphi_i}{\partial\Delta_i} = \frac{\Delta_i}{4}\left[C_i f_i(R_i) - b_{oi}f_i'(R_i)\right].$$

Since this term cannot be zero for all R, we get

$$\frac{Et}{p_i}\frac{\partial\lambda_i'}{\partial R_i} + \frac{\partial D(P)}{\partial P} = 0.$$

The system solution is then expressed by the following set of equations:

$$\frac{Et}{p_i}\frac{\partial\lambda_i'}{\partial R_i} + \frac{\partial D(P)}{\partial P} = 0$$

(33c)

$$\frac{\partial\lambda_i'}{\partial R_i} - 2\frac{\partial\lambda_i'}{\partial\Delta_i} = 0.$$

$$\sum_{i=1}^{n} (R_i + \frac{\Delta_i}{2}) - EP + \sum_{i=1}^{n} (\frac{1}{2} - \mathcal{E}_i) + E(\ell - S_o) = 0$$

If we now wish to re-introduce the shadow price d we observe, from equation (6), that

$$\left[b_{oi} f_i(R_i) - C_i \frac{\partial}{\partial R_i} \int_{R_i}^{\infty} (x - R_i) dF_i(x)\right] = \frac{\Delta_i d_i}{\mathcal{E}_i},$$

Substituting this into our result for $\frac{\partial \lambda_i'}{\partial R_i}$ we see that

$$\frac{\partial \lambda_i'}{\partial R_i} = - d_i$$

which must hold for all bases. Therefore, the first of equations (33c) becomes

$$\frac{d_i}{p_i} = \frac{1}{Et} \frac{\partial D(P)}{\partial P} = d.$$

The first of equations (26), when combined with our result for $\frac{\partial \lambda_i'}{\partial \Delta_i}$, shows that

$$\frac{\partial \lambda_i'}{\partial \Delta_i} = - \frac{d_i}{2},$$

so that the second of equations (33c) is no longer independent. However, in the third of equations (33c), Δ_i and R_i may be expressed as functions of d_i by equations (26). Therefore, we obtain the equations

$$d_i = \frac{p_i}{Et} \frac{\partial D(P)}{\partial P}$$

(33d)

$$\sum_{i=1}^{n} \left(R_i(d_i) + \frac{\Delta_i(d_i)}{2}\right) - EP + \sum_{i=1}^{n} (\frac{1}{2} - \mathcal{E}_i) + E(\ell - S_o) = 0$$

which may be solved simultaneously for d_i and P, where $R_i(d_i)$ and $\Delta_i(d_i)$ are given by equations (26).

Equations (33d) represent an easier set of equations than equations (33) for finding the optimal procurement amount, P. A computational method might be to assume a value for P, find the corresponding values for d_i from the first of equations (33d), and calculate optimal Δ_i and R_i for all bases, using this value for d_i in equations (26). When these optimal values are placed into the second of equations (33d), together with the assumed value for P, a result is obtained which may or may not be zero. If different from zero, the procedure is applied for different values of P until a P^* is obtained such that the result is nearest zero. This P^* represents the optimal procurement amount.

Equations (33d) provide a value for the procurement amount P, which is "optimal" in the sense that, given E, it provides least expected system costs. Of course, there may be some particular E which yields lower costs than any other value for E. On the other hand, we recognize the usefulness of assigning E to conform with military policies. In a sense, then, we obtain a sub-optimization for the system requirements calculation.

12. BASE STOCKAGE DISTRIBUTIONS

Having determined the amount to be procured, we now look at the problem of distributing the materiel among the bases and depots. Of the several ways in which this problem can be approached, we consider a method of recalculating base stockage levels at a number of times throughout the procurement period.

Let the procurement period be divided into m intervals by the points $t_1, t_2, \ldots t_{m+1}$, where t_1 is the beginning of the procurement period and t_{m+1} the end. At each time of calculation, t_i (i = 1 to m), we establish the ratio,

$$(34) \qquad E = \frac{\sum_{j=1}^{n} \left[R_j(d) - \ell_j + \frac{\Delta_j(d) + 1}{2} \right]}{S_i - \ell_i}$$

where the numerator is the expected base stockage at the end of the procurement period, S_i is the total system stocks at t_i, the time of calculation, ℓ_i is the expected consumption during the rest of the procurement period, and E is the same constant used in the previous section. Various values for the constant ℓ_i will be given in section 13. Equation (34) may be solved uniquely for d. This provides values for R_j and Δ_j ($j = 1$ to n for n bases) and determines the base stockages.

The distribution of amounts procured to the various depots is determined at time t_1. The procurement amount is distributed so that the resulting depot stocks are in the same ratio as the expected mean demands of the bases normally served by the respective depots. The mean demand, of course, must be determined in terms of some common unit of time. If there are two depots A and B, for example, then the respective stocks S_A and S_B after distribution of the procurement amounts should satisfy the relation

$$\frac{S_A}{M_A} = \frac{S_B}{M_B}$$

where M_A and M_B are the expected mean demands of bases served by depots A and B respectively.

Let us now look at the effect of the stockage distribution method described above. If part way through the procurement period, the demand for the item has been greater than anticipated, the expected system stocks at the end of the procurement period, $S_i - \ell_i$, will decline. Therefore, in order to maintain the ratio E, the shadow price d will increase. Thus, all base reorder levels and stock control levels will decrease and a uniform

stringency will be imposed on all bases. Base stocks will reach the reorder levels later and when they do, smaller orders will be sent. By thinning out stocks at all bases, this procedure reduces the probability of having to backorder any of them.

If, on the other hand, the demand for the item has been less than anticipated, the expected system stocks at the end of the procurement period will rise, the shadow price will fall, and all reorder levels and stock control levels will rise. Thus, base stocks will reach the reorder levels sooner and larger orders will be sent. In this way, the extra amounts will be spread throughout the system.

13. THE PROCUREMENT COST FUNCTIONS FOR EXPENDABLE, NON-RECOVERABLE ITEMS

In this section we consider different procurement cost functions which correspond to the different kinds of procurement policies for non-recoverable items. In this section, we will express the cost functions in terms of S rather than in the procurement amount P. Since S and P differ by the constant, S_o, this becomes just a matter of convenience.

First, we will define some of the terms to be commonly used in this section.

(1) K_s = fixed cost of procurement, including contracting costs and all procurement costs which do not vary with S. For open contract procurement, this cost is the fixed paperwork cost associated with placing an order on the factory.

(2) $U(P)$ = unit procurement cost of the items on a delivered basis. This function includes factory to depot packaging, inspection, and transportation costs. When $P = 0$, $U(0)$ is the initial set-up costs of manufacture.

(3) c = scrap value of the items or the salvage value, whichever is higher.

(4) p_s = routine pipeline time, for open contract procurement. It is defined as the number of days from initiation of an order on the factory to the time of arrival of the materiel in the depot. It differs from lead time, as previously defined, in that it contains no contracting time and usually little or no set-up time at the factory.

(5) ρ_s = expedited pipeline time for the relevant procurement. Its definition is the same as for p_s but uses premium communication and transportation.

(6) $f_i(x)$ = system demand probability functions where different values of i (i = 1, 2, 3) refer to the different procurement policies. Thus, $f_1(x)$ represents the demand probability for the remaining life of the item as used in the life-of-type procurement policy, $f_2(x)$ represents the demand probability over the consumption period as used in periodic procurement, and $f_3(x)$ represents the demand probability per p_s, the routine pipeline time from the manufacturer to the depot. This last function is used in the open contract procurement policy.

(7) $F_i(x) = \int_0^x f_i(t)dt$ = cumulative system demand probability function (i = 1, 2, 3,) corresponding to the three procurement policies.

(8) $\mathcal{E}_i = \int_0^\infty x f_i(x)dx$ = expected mean demand (i = 1, 2, 3) corresponding to the three procurement policies.

(9) ${}_s d_i + {}_s e_i y$ = cost of holding an average amount y, where the subscript i (i=1, 2, 3) refers to the three procurement policies. Thus, when i = 1, this cost represents the

holding cost over the remaining life of the item. When $i = 2$, it is the holding cost over the consumption period, and when $i = 3$, it is the holding cost per ρ_s. These costs include interest, warehousing, and all depreciation costs except obsolescence.

(10) ${}_s b_o + {}_s b_1 q + {}_s b_2 \rho_s q$ = system depletion penalty, or the cost of insufficient procurement. In this cost, ${}_s b_o$ is the fixed cost, if any, which is independent of the amount of depletion. The cost ${}_s b_1$ is the cost associated with the amount of depletion q, but not duration of depletion. The component ${}_s b_2 \rho_s q$ is the cost associated with the amount and duration of depletion. Here, ${}_s b_2$ could be the cost per consumption period of having an extra end item available for use. The term ρ_s is the duration of depletion, in terms of a fraction of the consumption period, while q is the amount of depletion. The various components of the depletion penalty will be discussed in further detail later.

A. Life-Of-Type Procurement

The procurement cost function, $D(S)$ for a life-of-type procurement policy and for a non-recoverable item is given by

$$(35) \qquad D(S) = K_s + U(0) + [U(S - S_o) - c] \int_0^{S^-} (S - x)\, dF_1(x)$$

$$+ {}_s d_1 + {}_s e_1 \left(S - \frac{\mathcal{E}_1 - 1}{2} \right) + [U(0) + K_s]\,[1 - F_1(S)] .$$

The second term $[U(S - S_o) - c] \int_0^{S^-} (S - x)\, dF_1(x)$, in the right member of equation (35), represents the expected obsolescence cost of procuring P items. In this term, we assess obsolescence of an item procured in accordance with the probability of that item not being used. If x items are demanded during the life of the item, and there are S items in the system initially, then $S - x$ items are not demanded, $(x \leq S)$. We multiply this number of items by the unit cost of the item, $[U(S - S_o) - c]$ and by the probability of demanding x items $dF_1(x)$. We then sum, or integrate, for values of x ranging from 0 to S^-, where the symbol S^- means integration to S but not including S. It might be noticed that we are pricing the amounts S_o which are on hand at the time of calculation at the same unit cost as the amounts procured.

The term $S - \frac{\mathcal{E}_1 - 1}{2}$ in equation (35) represents the average expected amount on hand during the life of the item. Its derivation is similar to the derivation of equation (3). This term is multiplied by ${}_s e_1$ the component of the lifetime holding costs associated with the amount held.

The last term of equation (35) represents the depletion penalty, where $U(0) + K_s$ is the cost of depletion and $1 - F_1(S)$ is the probability of depletion. The depletion penalty expressed in this way assumes that the depletion is predictable sufficiently in advance to obtain more materiel from the factory before actual depletion occurs. Therefore, we limit depletion costs to the fixed costs of procurement plus initial set-up costs at the factory.

If such additional procurement is made, then equation (35) is used again to find the amount of procurement. Of couse, the function $F_1(x)$ will be different in this case, since the expected remaining lifetime of the item has changed.

The value of ℓ for use in equation (33) is $\mathcal{E}_1$. To find the value of ℓ_i to use in equation (34) we suppose that $\mathcal{E}_{1i}$ is given for i = 1 to m, where $\mathcal{E}_{1i}$ is the expected average system demands during the i-th calculation period, and there are m such periods in the expected life of the item. If this is the case, $\mathcal{E}_1 = \ell = \sum_{i=1}^{n} \mathcal{E}_{1i}$ and $\ell_i = \sum_{k=i}^{m} \mathcal{E}_{1k}$. If the expected demand during any one calculation period is assumed the same as for all others, then $\ell_i = \dfrac{\mathcal{E}_1(m-i)}{m}$. The value for t for use in equations (33) is the expected life of the item, expressed in days.

B. Periodic Procurement

The procurement cost function D(S) for a periodic procurement policy and for a non-recoverable item is given by

$$(36)\qquad D(S) = K_s + U(0) + [U(S - S_o) - c] \int_0^{S^-} (S - x)\, dF_1(x)$$

$$+ {}_sd_2 + {}_se_2 \left(S - \frac{\mathcal{E}_2 - 1}{2}\right) + {}_sb_o [1 - F_2(S)]$$

$$+ {}_sb_1 \int_S^{\infty} (x - S)\, dF_2(x) + {}_sb_2 \int_S^{\infty} \frac{(x - S)(x + 1 - S)}{2(x + 1)}\, dF_2(x).$$

This equation is similar to equation (35) except for the last three terms which represent the depletion penalty. The term ${}_sb_o[1 - F_2(S)]$ represents the costs of depletion which are invariant with the quantity of depletion and the length of depletion, where ${}_sb_o$ is the cost and $[1 - F_2(S)]$ the probability of incurring the cost. The term ${}_sb_1 \int_S^{\infty} (x - S)\, dF_2(x)$ represents

the costs of depletion which vary with the amount of depletion but not with the length of depletion. In this term ${}_sb_1$ is the cost per depletion, $x - S$ is the amount of depletion and $dF_2(x)$ is the probability of demanding x items and incurring a depletion of $x - S$ items. The integration sums, for values of x from S to ∞, the products of amounts of depletion by probability of incurring that amount of depletion.

The term ${}_sb_2 \int_S^{\infty} \frac{(x - S)\,(x + 1 - S)}{2\,(x + 1)}\, dF_2(x)$ represents the costs of depletion which vary with the quantity of depletion and the length of depletion. In this term, ${}_sb_2$ is the per unit cost of such depletion, $(x - S)$ is the amount of depletion, $\frac{x + 1 - S}{2\,(x + 1)}$ is the expected duration of depletion, and $dF_2(x)$ is the probability of depletion by such an amount. These expected costs for all possible depletion amounts are summed by integrating over values of x from S to ∞.

The depletion penalty defined above assumes that the item cannot be recontracted for before the established contracting date if the system prematurely runs out of the item. However, if we do assume that additional quantities may be contracted for before the regular contracting date, then the depletion penalty becomes $[K_s + U(0)]\,[1 - F_2(S)]$, as was the case in life-of-type procurement. This assumes, of course, that the system is able to predict the impending depletion and obtain more materiel before actual depletion occurs.

The value for $\mathcal{E}$ for use in equations (33) is $\mathcal{E}_2$. As in the case of life-of-type procurement, the value for $\mathcal{E}_1$ to use in equation (34) is given by $\mathcal{E}_1 = \sum_{k=1}^{n} \mathcal{E}_{2k}$, where $\mathcal{E}_{2k}$ is the expected mean demand during the k-th calculation period, there being n such periods in the procurement period. The value of t for use in equations (33) is the procurement period, expressed in days.

C. Open Contract and Variable Period Procurement

For the open contract procurement policy, we have the procurement cost function expressed in terms of two variables rather than the single variable as in the other policies. These two variables are the system reorder level R_s and the system reorder amount (amount procured), Δ_s. The procurement cost equation is very similar to equation (5) for the base costs. The same terminology and definitions will therefore be used but prefexing s as an index to the various costs to distinguish clearly the two cases. The procurement cost function for open contract procurement of a non-recoverable item then becomes

$$(37) \qquad D(R_s, \Delta_s) = A_s + \frac{\mathcal{E}_3(B_s + {}_sb_o)}{\Delta_s} + \frac{{}_se_3}{2}\Delta_s + {}_se_3R_s - \frac{\mathcal{E}_{3\,s}b_o}{\Delta_s}F_3(R_s) + \frac{\mathcal{E}_3C_s}{\Delta_s}\int_{R_s}^{\infty}(x - R_s)\,dF_3(x)$$

where
$$A_s = {}_sd_3 + \mathcal{E}_3({}_sp_2 + {}_st_2) + {}_se_3\left(\frac{1}{2} - \mathcal{E}_3\right)$$

$$B_s = K_s + {}_sp_1 + {}_st_1$$

$$C_s = {}_sb_1 + {}_sb_2\rho_s$$

To solve for optimal R_s and Δ_s we must develop equations to replace equations (33). An equation analogous to equation (30) may be expressed as follows:

$$(38) \qquad L(R_s, \Delta_s, d\ ) = D(R_s, \Delta_s) + \lambda_s(d\ ).$$

This equation is subject to the restriction of equation (29), where $S_e = R_s - \ell$.

Using the method of section 11, we form the function

$$\Gamma(R_s, \Delta_s, R_i, \Delta_i) = L(R_s, \Delta_s, R_i, \Delta_i) + \mu_i\varphi_i(\Delta_i, R_i) + \mu\psi(R_s, \Delta_i, R_i),$$

and solve the following set of equations simultaneously for R_s, Δ_s, R_i, and Δ_i:

$$\frac{\partial \Gamma}{\partial \Delta_s} = \frac{\partial D}{\partial \Delta_s} = 0$$

$$\frac{\partial \Gamma}{\partial R_s} = \frac{\partial D}{\partial R_s} - \mu E = 0$$

$$\frac{\partial \Gamma}{\partial R_i} = \frac{t}{p_i} \frac{\partial \lambda'_i}{\partial R_i} + \mu_i \frac{\partial \varphi_i}{\partial R_i} + \mu = 0$$

$$\frac{\partial \Gamma}{\partial \Delta_i} = \frac{t}{p_i} \frac{\partial \lambda'_i}{\partial \Delta_i} + \mu_i \frac{\partial \varphi_i}{\partial \Delta_i} + \frac{\mu}{2} = 0$$

$$\varphi_i(\Delta_i, R_i) = 0$$

$$\psi(R_s, \Delta_i, R_i) = 0.$$

Using the results of section 11, we obtain the equations

$$d_i = \frac{p_i}{Et} \frac{\partial D(R_s, \Delta_s)}{\partial R_s}$$

(39)
$$\sum_{i=1}^{n} \left[R_i(d_i) + \frac{\Delta_i(d_i)}{2} \right] - ER_s + \sum_{i=1}^{n} \left(\frac{1}{2} - \mathcal{E}\right) + E(\ell - S_o) = 0$$

$$\frac{\partial D}{\partial \Delta_s} = 0$$

which may be solved simultaneously for d_i, R_s, and Δ_s. Of course, $R_i(d_i)$ and $\Delta_i(d_i)$ are given by equations (26).

The value for ℓ in equations (39) is $\mathcal{E}_3$. The value for t in equations (39) is p_s. It is noticed that the process of section 12 is not particularly applicable in the case of open contract procurement. The value for d which determines base stockage is obtained from equations (39), and does not change

unless from a recalculation of equations (39).

Equation (37) does not contain obsolescence costs as did the equations for the other procurement policies. This is under the assumption that the open contract policy would not be used when the item approaches the end of its life. However, if we are willing to recalculate optimal R_s and Δ_s each time the reorder level is reached, we may introduce obsolescence by adding the following term to equation (37):

$$\frac{\mathcal{E}_3\left[U(\Delta_s) - c\right]}{\Delta_s} \int_0^{R_s + \Delta_s} (R_s + \Delta_s - x)\, dF_1(x)$$

which is as previously defined except for dividing by $\frac{\Delta_s}{\mathcal{E}_3} = \overline{\theta}_s$ to express the cost per routine pipeline time, p_s.

14. THE GENERAL SYSTEM SOLUTION FOR RECOVERABLE ITEMS

The general system solution for recoverable items is similar in concept to the solution for non-recoverable items presented in section 11. Modifications must be made, however, to allow for items being in repair and hence unavailable for supply. Also, it is necessary to allow for gains to the system from attritted end items and for losses due to condemnations.

Before introducing the system cost equation for recoverable items, further notation must be developed. This notation will also be used in section 15.

(1) t = length of the consumption period, expressed as a number of units of time. The unit of time must be significantly large; in the order of a month. If the unit of time is a month, then there are t months in the consumption period.

(2) t' = the remaining lifetime of the item, measured in the unit of time.

(3) $\frac{\lambda'_{ij}}{p_i}$ = base stockage cost for the i-th base (i=1 to n) during the j-th unit of time (j = 1 to t), where p_i is the routine pipeline time of the i-th base expressed in terms of the unit of time.

(4) J_j = expected excess transportation costs during the j-th unit of time (j=1 to t).

(5) r_j = repair cycle ending with the j-th unit of time (j=1 to t). Here, r_j is expressed in terms of the unit of time.

(6) $f_{4j}(x)$ = system demand probability function over repair cycle r_j.

(7) $F_{4j}(x) = \int_0^x f_{4j}(t)dt$ = cumulative system demand probability function over repair cycle r_j.

(8) $\mathcal{E}_{4j}$ = mean expected system demands during the j-th unit of time (not the j-th repair cycle).

(9) $\bar{w}$ = average wearout rate, or the ratio of items condemned to total exchanged items. It is used below as the probability of an item received in exchange being condemned.

(10) w = number of items condemned; a random variable.

(11) $\phi_j(w)$ = the condemnation probability function for time periods from the time of computation up to but not including repair cycle r_j.

$= \int_w^\infty \psi(x,w)\, d\bar{F}_j(x)$, where $\psi(x,w)$ is the probability of condemning w items when x items are demanded $(w \leq x)$, and $\bar{F}_j(x)$ is the cumulative demand probability function for time periods from the time of computation up to but not including repair cycle r_j. It can be shown that if x takes only integral values, then $\psi(x,w) = (1 - \bar{w})^{x-w}$ $\bar{w}^w \dfrac{x!}{w!\,(x-w)!}$ [1].

1/ See J. V. Uspensky, INTRODUCTION TO MATHEMATICAL PROBABILITY, p 46.

(12) $\Phi_j(w) = \int_0^w \phi_j(t)dt$ = cumulative condemnation probability function for time periods from the time of computation up to but not including repair cycle r_j.

(13) $\phi(w) = \int_w^{\infty} \psi(x,w)\, dF_1(x)$ = the lifetime condemnation probability function.

(14) $\Phi(w) = \int_0^w \phi(t)dt$ = cumulative lifetime condemnation probability function.

(15) $\Phi'(w)$ = cumulative condemnation probability function for the consumption period.

(16) a_j = expected total number of items (assumed reparable) obtained from end items which are attritted in the j-th unit of time. It does not include condemned items from attritted end items. Each a_j is assumed constant.

(17) $S_j' = S + \sum_{k=1}^{j-r_j} a_k$ = expected amount of serviceables, reparables and condemned items in the system at the beginning of the j-th repair cycle.

(18) $v(S)$ = average expected value of an item in a reparable condition. This value may be given as the procurement unit cost $\mho(S)$, minus average cost of repairing the items minus average transportation costs from base to depot. The value $v(S) - c$ is always positive since if the scrap value is greater than v, the item is condemned rather than repaired.

(19) p_s' = procurement and contracting lead time measured in units of time.

An equation analogous to equation (28) may now be expressed for the case

of the recoverable type item as follows:

$$(40) \qquad L(P) = D(P) + \sum_{j=p_s'+1}^{t} \int_{w=0}^{S_j'} \int_{x=0}^{S_j'-w} M_j(P,x,w)\, dF_{4j}(x)\, d\,\Phi_j(w)$$

$$+ \sum_{j=p_s'+1}^{t} \int_{w=0}^{S_j'} M_j(P,x=S_j'-w,w) \left[1 - F_{4j}(S_j'-w)\right] d\Phi_j(w)$$

$$+ \sum_{j=p_s'+1}^{t} M_j(P,x=0,w=S_j') \left[1 - \Phi_j(S_j')\right]$$

where
$$M_j(P,x,w) = \sum_{i=1}^{n} t \, \frac{\lambda_{ij}'(P,x,w)}{p_i} + J_j(P,x,w)$$

Equation (40) differs from equation (28) by the addition of the random variables, w and x, which represent the amount condemned and the amount in repair, respectively. The function M_j is only implicitly a function of P, x, and w; in particular, λ_{ij}' is explicitly a function of R_i and Δ_i. Of course, equation (40) is restricted by the condition that total expected base stocks do not exceed total expected system stocks. This restriction is expressed by equation (41) below, where E is set equal to 1. Different functions for $D(P)$ in equation (40) are developed in section 15 to correspond to the different procurement policies.

In equation (40), the functions J_j are impossible to obtain, so we again resort to a sub-optimization by use of a fixed ratio of base stocks to system stock. Here, however, we impose this ratio for each unit of time in the consumption period. Therefore, we obtain the equations

$$(41) \qquad E = E_j = \frac{\sum_{i=1}^{n} \left[R_{ij} + \frac{\Delta_{ij}+1}{2} - \varepsilon_{ij}\right]}{S_j' - x - w}$$

By a method analogous to that of section 11, we obtain the optimal procurement amount, P, as a solution of the equation:

$$(42)\qquad \sum_{j=p_s'+1}^{t} \int_{w=0}^{S_j'} \int_{x=0}^{S_j'-w} d_{ij}(P,x,w)\ dF_{4j}(x)\ d\Phi_j(w)$$

$$+ \sum_{j=p_s'+1}^{t} \int_{w=0}^{S_j'} d_{ij}(P,x=S_j'-w,w)\ \left[1 - F_{4j}(S_j'-w)\right]\ d\Phi_j(w)$$

$$+ \sum_{j=p_s'+1}^{t} d_{ij}\ (P,x=0,w=S_j')\left[1 - \Phi_j(S_j')\right] = \frac{p_i}{tE}\ \frac{\partial D(P)}{\partial P}$$

where $d_{ij}(P,x,w)$ are given as solutions of the equations

$$\sum_{i=1}^{n}\left[R_{ij}(d_{ij}) + \frac{\Delta_{ij}(d_{ij})}{2}\right] + \sum_{i=1}^{n}\left(\frac{1}{2} - \epsilon_{ij}\right) - E_jS_j' + E_j(x+w) = 0.$$

Of course, in the above equations, $R_{ij}(d_{ij})$ and $\Delta_{ij}(d_{ij})$ are solutions of equations (8) and (9) expressed in terms of the i-th base and j-th period. Also, it should be remembered that $S_j' = P + S_o + \sum_{k=1}^{j-r_j} a_k$.

The above results are used only to find the procurement amount P, and are not used for subsequent base stockage distributions. The discussion of section 12 still applies for the base stockage distributions, except that equation (41) is used instead of equation (34).

15. THE PROCUREMENT COST FUNCTIONS FOR EXPENDABLE, RECOVERABLE ITEMS

In this section, we consider different procurement cost functions which correspond to the different kinds of procurement policies for recoverable items.

A. Life-of-Type Procurement

The procurement cost function D(S) for a life-of-type procurement policy

and for a recoverable item is given by

$$(43)\qquad D(S) = K_s + U(0) + \left[v(S-S_o) - c\right]\left[(1 - \bar{w} \sum_{k=t'-r_{t'}+1}^{t'} \mathcal{E}_{4k} + \int_{o}^{S_{t'}} (S_{t'} - w)\, d\,\Phi(w)\right]$$

$$+ {}_s d_1 + {}_s e_1 \left[S + \frac{\sum_{k=1}^{t'} a_k - \bar{w} \sum_{k=1}^{t'} \mathcal{E}_{4k} + 1}{2} \right]$$

$$+ \left(K_s + U(0)\right)\left[1 - \Phi\left(S + \sum_{k=1}^{t} a_k - (1 - \bar{w}) \sum_{k=t-r_t+1}^{t} \mathcal{E}_{4k}\right)\right]$$

$$+ {}_s b_o \sum_{j=p_s'+1}^{t} \int_{o}^{S_j'} \left[1 - F_{4j}(S_j' - w)\right] d\,\Phi_j(w)$$

$$+ {}_s b_1 \sum_{j=p_s'+1}^{t} \int_{o}^{S_j'} \int_{S_j'-w}^{\infty} (x - S_j' + w)\, dF_{4j}(x)\;\; d\,\Phi_j(w)$$

$$+ {}_s b_2 \sum_{j=p_s'+1}^{t} \int_{o}^{S_j'} \int_{S_j'-w}^{\infty} \frac{(x-S_j'+w)\;(x+1-S_j'+w)}{2(x+1)}\, dF_{4j}(x)\; d\Phi_j(w)$$

where $S_{t'} = S + \sum_{k=1}^{t'} a_k - \sum_{k=t'-r_{t'}+1}^{t'} \mathcal{E}_{4k}$

In equation (43) the second term in the right hand side of the equation:

$$\left[v(S-S_o) - c\right]\left[(1-\bar{w}) \sum_{k=t'-r_{t'}+1}^{t'} \mathcal{E}_{4k} + \int_{o}^{S_{t'}} (S_{t'}-w)\, d\,\Phi(w)\right]$$

represents the cost of obsolescence. In this term. $[v(S-S_o) - c]$ represents the cost of one item becoming obsolete. By using $[v(S-S_o) - c]$ rather than $[U(S-S_o) - c]$ we assume that the system is able to predict the remaining demand, at some point in time close to the end of the life of the item, sufficiently well to halt repair on items that are not going to be issued. Thus, the system saves the cost of repairing and transporting the reparable which is never going to be issued. We feel that if $[v(S-S_o) - c]$ is not an exact statement of the loss due to obsolescence on a recoverable item, it is at any rate preferable to $[U(S-S_o) - c]$.

The term $(1-\bar{w}) \sum_{k=t'-r_{t'}+1}^{t'} \varepsilon_{4k}$ represents the number of items that must become obsolete. It is the number expected to be issued in the last repair cycle, minus those that will be condemned in that time period. This amount must become obsolete even if it is necessary to procure more of the item to compensate.

The term $S_{t'}$ represents the amount of stock that must be condemned in order to avoid all obsolescence on the item, except that amount which cannot be avoided, as described in the preceding paragraph.

The term $\int_0^{S_{t'}} (S_{t'}-w)\ d\Phi(w)$ represents the expected number of items becoming obsolete. The term $S_{t'}-w$ represents the number of items becoming obsolete, given w condemnations; $d\Phi(w)$ is the probability of w condemnations in the lifetime of the item, and the product $(S_{t'}-w)\ d\Phi(w)$ represents the product of the number of items becoming obsolete, and the probability of incurring that number of condemnations. The integral sums these products from $w = 0$, in which case the whole of $S_{t'}$ becomes obsolete, to $S_{t'}$, in which case none of $S_{t'}$ becomes obsolete.

The fifth term in the right hand side of the equation:

$$ {}_{s}e_{1} \left[S + \frac{\sum_{k=1}^{t'} a_k - \overline{w} \sum_{k=1}^{t'} \varepsilon_{4k} + 1}{2} \right] $$

represents the holding costs, excluding obsolescence, where the bracketed part is the average expected amount of the item on hand over the lifetime of the item.

The sixth term:

$$ (K_s + U(0)) \left[1 - \Phi \left(S + \sum_{k=1}^{t} a_k - (1-\overline{w}) \sum_{k=t-r_t+1}^{t} \varepsilon_{4k} \right) \right] $$

represents the expected cost of having to procure a second time. The term $(K_s + U(0))$ is the cost of having to procure a second time;

$$ \left[1 - \Phi \left(S + \sum_{k=1}^{t} a_k - (1-\overline{w}) \sum_{k=t-r_t+1}^{t} \varepsilon_{4k} \right) \right] $$

represents the probability of condemning more than $S + \sum_{k=1}^{t} a_k - (1-\overline{w}) \sum_{k=t-r_t+1}^{t} \varepsilon_{4k}$, which is the maximum amount of stock that can be condemned without an additional procurement of stock. In assessing this cost, it is again assumed that additional procurement can be obtained before actual depletion of serviceables and reparables occurs.

The seventh term:

$$ {}_{s}b_{o} \sum_{j=p'_s+1}^{t} \int_{o}^{S'_j} \left[1 - F_{4j}(S'_j - w) \right] d\,\Phi_j(w) $$

represents the expected system depletion costs which do not vary with either the size of depletion or the length of depletion, resulting from a temporary pile-up of reparable items undergoing repair. The term ${}_{s}b_{o}$ represents the

cost of one such depletion; the term $[1 - F_{4j}(S_j^!-w)]$ represents the probability of exchanging more than $S_j^!-w$ during the repair cycle r_j; the integral sums for all values of w from 0 to $S_j^!$; and the summation sums the depletion probability for each repair cycle r_j in period t. Issues (or exchanges) are relevant in this term, since each item in the system at the beginning of the r_j-th repair cycle, whether serviceable or reparable at that time, is available for issue once, and only once, during the repair cycle. If the item is serviceable at the beginning of the period, it is clearly available for issue, but just as clearly, the item which it replaces on the end item, whether condemned or reparable, cannot be repaired before the end of the repair cycle, and so cannot be issued during the period. If the item is a reparable at the beginning of the repair cycle, it can be repaired and issued before the end of the repair period, and will be available for issue once; the item it replaces cannot be made serviceable in time for issue in that repair period. The amount $S_j^!-w$ represents the stock of serviceables and reparables on hand at the beginning of the j-th repair period.

The summation $\sum_{j=p_s^!+1}^{t}$ represents the summation of independent probabilities of depletion, one for each repair cycle r_j. Since r_j is probably larger than the unit of time, each unit of time may be in several repair cycles. The probabilities of depletion in each repair cycle, despite the overlap, may be assumed independent if the unit of time is large enough so that the probability of depletion in one repair cycle, in the absence of one of the units of time therein, is negligible. It is therefore recommended that three units of time be the maximum in any one repair cycle. For this reason, the month has been suggested as the unit of time. The lower limit of the summation is $p_s^!+1$ since the probability of depletions occurring

prior to p'_s+1 cannot be affected by the current procurement decision.

The eighth term in equation (43):

$$ {}_s b_1 \sum_{j=p'_s+1}^{t} \int_0^{S'_j} \int_{S'_j-w}^{\infty} (x - S'_j + w)\, dF_{4j}(x)\ \ d\Phi_j(w) $$

represents the expected system depletion costs which vary with the size of depletion but not with the duration of depletion, resulting from a temporary pile-up of reparable items undergoing repair. Here ${}_s b_1$ is the cost of the depletion per unit of depletion; $(x-S'_j+w)$ is the size of the depletion, given a quantity of x issues in repair period r_j; $dF_{4j}(x)$ is the probability of x issues in repair period r_j; the second integral sums the products of each size of depletion and probability of that size of depletion from $x = S'_j-w$ to infinity; and the first integral sums the products of the cost, given w, and the probability of w for values from 0 to S'_j. The summation is the same as the summation in the seventh term.

The ninth and last term in equation (43)

$$ {}_s b_2 \sum_{j=p'_s+1}^{t} \int_0^{S'_j} \int_{S'_j-w}^{\infty} \frac{(x-S'_j+w)\ (x+1-S'_j+w)}{2(x+1)}\ dF_{4j}(x)\ d\Phi_j(x), $$

represents the expected system depletion costs which vary with both the size of depletion and the duration of depletion, resulting from a temporary pile-up of reparable items undergoing repair. The term ${}_s b_2$ is the cost per unit of depletion of a depletion lasting throughout the repair cycle; the term $\frac{x+1-S'_j+w}{2(x+1)}$ is the expected average duration, in terms of a fraction of the repair cycle, of a depletion of an amount $(x-S'_j+w)$; the term $(x-S'_j+w)$ is the amount of depletion given x; and $dF_{4j}(x)$ is the probability of occurring a depletion of size $(x-S'_j+w)$. The second integral sums the products of amount

of depletion, duration of depletion, and probability of that amount of depletion for all x from $S_j'-w$ to infinity. The first integral and the summation are the same as in the seventh term.

B. Periodic Procurement

The procurement cost function $D(S)$, for a periodic procurement policy, and for a recoverable item is given by:

$$(44) \quad D(S) = K_s + U(0) + \left[v(S-S_o) - c\right]\left[(1-\bar{w}) \sum_{k=t'-r_{t'}+1}^{t'} \mathcal{E}_{4k} + \int_o^{S_{t'}} (S_{t'}-w) \, d\Phi(w)\right]$$

$$+ {}_sd_2 + {}_se_2 \left[S + \frac{\sum_{k=1}^{t} a_k - \bar{w} \sum_{k=1}^{t} \mathcal{E}_{4k} + 1}{2}\right]$$

$$+ (K_s + U(0)) \left[1 - \Phi'\left(S + \sum_{k=1}^{t} a_k - (1-\bar{w}) \sum_{k=t-r_t+1}^{t} \mathcal{E}_{4k}\right)\right]$$

$$+ {}_sb_o \sum_{j=p_s'+1}^{t} \int_o^{S_j'} \left[1 - F_{4j}(S_j'-w)\right] d\Phi_j(w)$$

$$+ {}_sb_1 \sum_{j=p_s'+1}^{t} \int_o^{S_j'} \int_{S_j'-w}^{\infty} (x-S_j'+w) \, dF_{4j}(x) \, d\Phi_j(w)$$

$$+ {}_sb_2 \sum_{j=p_s'+1}^{t} \int_o^{S_j'} \int_{S_j'-w}^{\infty} \cdot \frac{(x-S_j'+w)(x+1-S_j'+w)}{2(x+1)} \, dF_{4j}(x) \, d\Phi_j(w).$$

The difference between equation (44) and equation (43) is that in equation (43), t', the remaining lifetime of the item, and t, the consumption period, are equal while they are not equal in equation (44). In this equation

for periodic procurement, we assume reprocurement before actual depletion occurs. However, reprocurement would occur only because of condemnations and not because of pile-ups in repair. Since the probability of condemnations is included in the procurement amount calculation, the probability of depletion through condemnations becomes relatively small.

C. Open Contract and Variable Period Procurement

For the open contract procurement policy, and variable period procurement policy, we have the procurement cost function expressed in terms of two variables rather than the single variable as in other policies. These two variables are the system reorder level R_s, and the system reorder amount Δ_s.

Before introducing the cost equations for the open contract and variable period procurement policy for a recoverable item, further notation must be developed.

(1) p_s = unit of time = period from the time system stocks (serviceables plus reparables) reach the system reorder level R_s to the time the materiel is received from the factory.

(2) $a = \frac{1}{t'} \sum_{j=1}^{t'} a_j$ = average attrition rate per unit of time p_s.

(3) $\mathcal{E}' = \mathcal{E}_3 \overline{w} - a$ = net expected losses per unit of time p_s.

(4) $\overline{\theta}_s = \frac{\Delta_s}{\mathcal{E}'}$ = average system procurement period, and is defined as the average period between successive deliveries to the system.

(5) $S''_j = R_s + \Delta_s + \sum_{k=1}^{j-r_j} a_k - \overline{w} \sum_{k=1}^{j-r_j} \mathcal{E}_{4k}$

(6) All other terms are as previously defined.

The total system cost equation, analogous to equation (40), may now be expressed as follows:

$$(45)\qquad L(R_s,\Delta_s) = D(R_s,\Delta_s) + \sum_{j=2}^{\frac{\Delta_s}{\mathcal{E}'}+1} \int_0^{S''_j} M_j(R_s,\Delta_s,x)\, dF_{4j}(x)$$

$$+ \sum_{j=2}^{\frac{\Delta_s}{\mathcal{E}'}+1} M_j(R_s,\Delta_s,x{=}S''_j)\ \left[1 - F_{4j}(S''_j)\right]$$

where $\quad M_j = \sum_{i=1}^{n} \lambda_{ij}(R_s,\Delta_s,x) + J_j(R_s,\Delta_s,x)$

Equation (45) may be solved for optimal R_s and Δ_s by setting the partial derivatives with respect to R_s and Δ_s equal to zero, and solving the two equations simultaneously for R_s and Δ_s, again with the restraint that expected base stocks do not exceed total system stocks.

Again, as in equation (41) we sub-optimize by replacing the J_j functions with the restriction that R_{ij} and Δ_{ij} satisfy the equations

$$E = E_j = \frac{\sum_{i=1}^{n} \left[R_{ij} + \frac{\Delta_{ij}+1}{2} - \mathcal{E}_{ij}\right]}{S''_j - x}$$

We may now write the procurement cost function $D(R_s,\Delta_s)$ as follows

$$(46)\qquad D(R_s,\Delta_s) = \frac{\mathcal{E}' K_s}{\Delta_s} + {}_sd_3 + {}_se_3 \left[R_s + \frac{\Delta_s+1}{2} - \mathcal{E}'\right]$$

$$+ \mathcal{E}'\,({}_sp_2 + {}_st_2) + \frac{\mathcal{E}'}{\Delta_s}\,({}_sp_1 + {}_st_1)$$

$$+ \frac{\mathcal{E}'\,{}_sb_o}{\Delta_s} \sum_{j=2}^{\frac{\Delta_s}{\mathcal{E}'}+1} \left[1 - F_{4j}(S''_j)\right] + \mathcal{E}'\,{}_sb_1 \sum_{j=2}^{\frac{\Delta_s}{\mathcal{E}'}+1} \int_{S''_j}^{\infty} (x-S''_j)\,dF_{4j}(x)$$

$$+ \frac{\mathcal{E}'\,{}_sb_2}{\Delta_s} \sum_{j=2}^{\frac{\Delta_s}{\mathcal{E}'}+1} \int_{S''_j}^{\infty} \frac{(x-S''_j)(x+1-S''_j)}{2(x+1)}\, dF_{4j}(x).$$

In equation (46) the first four terms are similar to the same terms in

equation (37); the last three terms are similar to the three depletion terms in equation (44), except that condemnations are treated as an expected mean value rather than as a random variable. We were able to make this simiplification in equation (46) because of the fact that the system reorder level is defined in terms of an amount of <u>uncondemned</u> items, and therefore, constrains the variation in the system stock of uncondemned items within limits. Thus, we may expect S''_j to attain each value between $R_s + \Delta_s - \bar{w}\mathcal{E}'$ and R_s once each ordering cycle, and may expect, by averaging over a large number of cycles, that S''_j will exist for a period of time equal to that which would be obtained by using an expected value for condemnations. We also assume that p_s is shorter than the repair cycle, so that any demand which occurs after R_s is reached will lead to a reduction in the stock of serviceables which is independent of whether the item received in exchange is condemned or reparable.

Note that the summation in the depletion terms is from 2 to $\bar{\theta}_s + 1$. The time lag of one period represents the procurement pipeline p_s; p_s is added to both the lower limit and the upper limit of the depletion term summations since the R_s and Δ_s to be calculated cannot affect the probability of depletion during the first period, but will affect, and indeed will determine the probability of depletion in period $\bar{\theta}_s + 1$ (or $\frac{\Delta_s}{\mathcal{E}'} + 1$).

If we are willing to recompute R_s and Δ_s each time the system reorder level, R_s is reached, we may add an obsolescence term to equation (46):

$$+ \left[v(S-S_o) - c\right] \left[(1-\bar{w}) \sum_{k=t'-r_{t'}+1}^{t'} \mathcal{E}_{4k} + \int_o^{\sigma} (\sigma - w)\, d\Phi(w)\right],$$

where $\sigma = R_s + \Delta_s + at' - \sum_{k=t'-r_{t'}+1}^{t'} \mathcal{E}_{4k}$.

16. FUNDS AND BUDGET

In computing the impact of the procurement requirements calculations on either available funds or budget requirements, the calculation is carried forth for as many periods t or $\bar{\theta}_s$ as are necessary to exhaust either the operating or the budget period. In the life-of-type calculations, the procurement amount must be multiplied by the unit cost to obtain the funds or budget impact. In the periodic procurement calculation, the procurement amount must be calculated for as many procurement periods as are wholly or partly in the operating or budget period. In each calculation, it is necessary to estimate the balance on hand at the time of the next calculation. This amount is $S'_{t-p'_s} - \bar{w} \sum_{k=1}^{t-p'_s} \mathcal{E}_{4k}$. In calculating the procurement requirements for the next period, $S'_{t-p'_s} - \bar{w} \sum_{k=1}^{t-p'_s} \mathcal{E}_{4k}$ is used as the beginning balance on hand. The total amount to be procured is then extended by the unit cost of procurement to obtain the impact on funds or budget requirements.

For the open-contract or variable period calculations, the impact on funds or budget requirements is $R_s + n \Delta_s - S_o$, where n is the number of periods $\bar{\theta}_s$ wholly or partly in the operating or budget period, and S_o is the current balance on hand.

If there is a fund limit within which all procurement requirements must fit, it is possible to sub-optimize for procurement of all items subject to the fund limit. Clearly, the fund limit must be considered a ceiling only; that is, if the dollar requirement for procurement of all items is less than the fund limit, it cannot be considered as either optimal or sub-optimal to raise procurement up to this limit. The problem before us consists of the necessity of forcing the dollar procurement requirement down to the fund limit, or, in other words, to sub-optimize procurement under constraint of the fund limit.

For this purpose, we may add the term $\Omega\, U(S)S$ to all life-of-type and periodic procurement equations; and the term $\Omega\, U(\Delta_s)(R_s + \Delta_s')$ to all open contract and variable procurement equations. The parameter Ω should be held at zero for the first calculation of the dollar requirement for procurement. If the total dollar procurement requirement is greater than the fund limit, Ω may then be raised in increments until the dollar procurement requirement is equal to the fund limit.

17. REFINEMENTS TO THE SYSTEM SOLUTION

A. The term ${}_sb_2$ as a Function of Amount of Depletion

In all previous formulations ${}_sb_2$, the coefficient of the cost of system depletion which varies with the quantity of depletion and the duration of depletion, has been assumed to be constant. We may wish to make ${}_sb_2$ a function of the amount of depletion. For example, the military organization may be able to prevent any loss of time in commission of the applicable end item, despite a depletion, through the device of "maintenance cannibalization"; in which the depleted item is removed from an end item entering maintenance and placed on an end-item which only needs the depleted part to become available for use. In this way, an end item out of commission is avoided by incurring the cost of one extra removal and one extra installation of the depleted spare item. It would not be appropriate to use this cost of an extra exchange as the depletion penalty since maintenance cannibalization is probably limited; it may suffice to prevent end items out of commission for the first ten depletions, but beyond that, depletions may cause end items to be out of commission. Therefore, ${}_sb_2$ becomes a function of the size of depletion.

We may introduce this modification into the system equations merely by moving ${}_sb_2$ inside the integral as a function of the amount of depletion

whenever it appears; that is, in equations (36), (37), (43), (44), and (46). For example, in equation (36), the last term would become:

$$\int_{S}^{\infty} {}_{S}b_2\,(x-S)\ \frac{(x-S)(x+1-S)}{2(x+1)}\, dF_2(x)$$

Here, ${}_{S}b_2(x-S)$ is the cost, which varies with the amount and duration of depletion, of $(x-S)$ depletions.

The same modification can be performed on b_2 in the base equations.

B. The Obsolescence Concept

Our treatment of obsolescence in the system equations is based upon the assumption that all end items will be phased-out as whole end items; that is, not lacking the spare item. Thus, the number of spare items condemned represents a fixed minimum requirement for the spare item. The logistics system is not charged the procurement cost of the condemned item since it is assumed not to be within its discretion whether that item is bought or not. It is, however, charged a holding cost if the item is bought too early, and a depletion cost if the item is bought too late. The assumption will be completely valid if, for example, the military organization wishes to mothball the applicable end item as a whole end item at the time of phase-out; or to present the end item to a friendly nation, again, as a whole end item.

If the military organization is willing to phase its end items out as incomplete end items, we may substitute the term:

$$U(S)S$$

for the obsolescence term in all system equations.

C. Variable End Item Maintenance Flow Time

The terms b_2 and ${}_{S}b_2$, whether constant coefficients, or variables, have been developed on the assumption that the maintenance flow time of the end item is given. If b_2 and ${}_{S}b_2$ are interpreted as the loss of utility

of the end item, they must be modified by the expectation that the end item requiring the part may have been out of commission already when the demand for the spare item arose. The end item may have been out of commission for another part, or, more likely, may have been out of commission because the need arose while the end item was undergoing overhaul. If, on the average, three depletions out of ten lead to end items out of commission and seven out of ten do not, for either of the foregoing reasons, b_2 and ${}_sb_2$ must be developed as .3 times the lost utility of the end item, rather than the whole lost utility of the end item. If the maintenance flow time of the end item were reduced, as a matter of policy, then we should expect that a larger per cent of depletions would lead to end items out of commission.

If maintenance cannibalization is used to measure the depletion penalty, the maintenance flow time of the end item becomes important in determining the number of exchanges in the spare part repair period of time in which ${}_sb_2$ is defined. For example, if the spare part repair period of time is 30 days, and the end item maintenance flow time is 10 days, a system depletion of the spare part lasting one period of time will be valued at the cost of three exchanges through cannibalization. If the maintenance flow time for the end item is then cut to 5 days, the same system depletion of the spare part will be valued at the cost of six exchanges.

Thus, b_2 and ${}_sb_2$, and also L, the system cost of logistics support occasioned by one spare item, become functions of the end item maintenance flow time.

Let us introduce some new terminology at this time:

(1) m_j = maintenance flow time of the applicable end item in period j.

(2) $L_i(m_j)$ = lifetime system logistics cost of spare item i as a function of m_j.

(3) t = total useful life of the end item.

(4) $M^1(m_j)$ = lifetime cost of a pool of end items out of commission for maintenance as a function of m_j.

(5) $M_j^2(m_j)$ = cost of maintenance on the end item in period j, including direct and indirect labor and capital, as a function of m_j.

(6) C_t = total lifetime cost of maintenance, logistics support, and a pool of end items out of commission for both maintenance and parts, over the lifetime of the end item.

We may now write an equation for C_t as:

$$C_t(m_j) = \sum_{i=1}^{n} L_i(m_j) + M^1(m_j) + \sum_{j=1}^{t} M_j^2(m_j) \tag{47}$$

where there are n spare items applicable to the end item.

We may then take t partial derivatives, one for each of the m_j; set them equal to zero; and solve simultaneously for the m_j.

D. The Batching Concept in Repair

In defining the repair cycle in period j as a constant, we have assumed that all reparable items are scheduled through repair as fast as they are generated (although the repair cycle may vary between periods). The military organization, however, may be faced with a large fixed cost per batch of items scheduled through repair. We present below an alternative formulation of the system cost equation which may be used for determining simultaneously the requirements from procurement and the optimal batch size for repair. We shall present the case of life-of-type procurement as an example.

Let us first introduce some new terminology:

(1) t' = the lifetime of the item measured in major periods of time (periods in the order of a year) t_m.

(2) t_m = the major period of time measured in minor periods of time, p_R.

(3) p_R = the minor period of time which is defined as the period from starting the batch of reparables through repair to their emergence as serviceables.

(4) Δ_{Rj} = the repair quantity or batch size in the j-th major period; ($\Delta_{Rj} \geq 1$).

(5) $S_{Rj} - w = S + \sum_{k=1}^{j-1} a_k + \frac{a_j}{2} - w$ = the average quantity of serviceables and reparables in the system in the j-th major period (assumed constant over the major period), where w, as before, represents condemnations and is a random variable.

(6) $S_{Rj} - \Delta_{Rj} - w$ = the repair level, or quantity of serviceables in the system at the time that repair on the batch is begun.

(7) $f_{Rj}(x)$ = the demand probability function over the minor period p_R in the j-th major period.

(8) $F_{Rj}(x) = \int_0^x f_{Rj}(t)dt$ = the cumulative demand probability function over minor period p_R in the j-th major period.

(9) $\bar{\theta}_{Rj} = \frac{\Delta_{Rj}}{\mathcal{E}_{Rj}}$ = the average cycle between receipts from repair in the j-th major period, where $\mathcal{E}_{Rj}$ is the mean of $f_{Rj}(x)$.

(10) $\Phi_j(w)$ = the cumulative condemnation probability function for periods prior to major period j, and for half of major period j.

(11) p_i = the routine depot-to-base pipeline for base i, measured in units of p_R.

(12) $d_{R1} + d_{R2}(y)$ = the holding cost over the major period, where y is the average amount of the item in the system in the major period

(13) $r_f + r_v(q)$ = the cost of repairing a batch of items, where q is the number of items in the batch

(14) $J_j(S_{Rj},x,w)$ = excess transportation costs defined over the minor period, p_R.

An equation analogous to equation (40) may now be expressed as follows:

$$(48) \qquad L(S,\Delta_{Rj}) = D(S,\Delta_{Rj}) + \sum_{j=p_s'+1}^{t'} \frac{t_m \mathcal{E}_{Rj}}{\Delta_{Rj}} \int_{w=0}^{S_{Rj}} \Bigg\{ \int_0^{\Delta_{Rj}} \frac{M_j(S_{Rj},x,w)}{\mathcal{E}_{Rj}} F_{Rj}(x)dx$$

$$+ \int_0^{(S_{Rj}-\Delta_{Rj}-w)^-} \frac{M_j(S_{Rj}-\Delta_{Rj},x,w)}{\mathcal{E}_{Rj}} \left[1 - F(x^-)\right] dx$$

$$+ \int_{S_{Rj}-\Delta_{Rj}-w}^{\infty} \frac{(x-S_{rj}+\Delta_{Rj}+w)}{x} M_j(S_{Rj},x=S_{Rj}-\Delta_{Rj}-w,w)\, dF(x) \Bigg\} d\Phi(w),$$

where $M_j(S_{Rj},x,w) = \sum_{i=1}^{n} \frac{\lambda'_{ij}(S_{Rj},x,w)}{p_i} + J_j(S_{Rj},x,w)$.

Of couse, this expression is subject to the restriction

$$E = E_j = \frac{\sum_{i=1}^{n} \left(R_{ij} + \frac{\Delta_{ij}+1}{2} - \mathcal{E}_{ij}\right)}{S_{Rj} - x - w} .$$

The system procurement function, $D(S,\Delta_{Rj})$ may be expressed as follows:

$$(49) \qquad D(S,\Delta_{Rj}) = \sum_{j=p_s'+1}^{t'} \int_{w=0}^{S_{Rj}} \Bigg\{ d_{R1} + d_{R2}(S_{Rj}-w) + \mathcal{E}_{Rj}\, t_m\, r_v$$

$$+ \frac{\mathcal{E}_{Rj} t_m}{\Delta_{Rj}} r_f + \frac{\mathcal{E}_{Rj} t_m}{\Delta_{Rj}} \Bigg[{}_s b_o \left[1 - F_{Rj}(S_{Rj} - \Delta_{Rj} - w)\right]$$

$$+ {}_s b_1 \int_{S_{Rj}-\Delta_{Rj}-w}^{\infty} (x-S_{Rj}+\Delta_{Rj}+w)\, dF_{Rj}(x)$$

$$+ \frac{b_s}{2} \int_{S_{Rj}-\Delta_{Rj}-w}^{\infty} \frac{(x-S_{Rj}+\Delta_{Rj}+w)(x+1-S_{Rj}+\Delta_{rj}+w)}{2(x+1)} \, dF_{Rj}(x) \Bigg] \Bigg\} d\Phi(w)$$

$$+ K_s + U(0) + [v(S-S_o) - c] \Bigg[(1-\bar{w}) \sum_{k=t'-r_{t'}+1}^{t'} \mathcal{E}_{4k}$$

$$+ \int_{0}^{S_{t'}} (S_{t'}-w) \, d\Phi(w) \Bigg]$$

$$+ [K_s + U(0)] \left[1 - \Phi \left(S + \sum_{k=1}^{t'} a_k - (1-\bar{w}) \sum_{k=t'-r_{t'}+1}^{t'} \mathcal{E}_{4k} \right) \right]$$

There are two rather questionable assumptions implicit in these equations. First, it is assumed that the wearout rate is low enough for S_{Rj} to be a reasonably close approximation to the stock of serviceables and reparables throughout major period j. Second, it is assumed that the probability of $x > \Delta_{Rj}$ is negligible. The formulation could, however, be refined, and the first of these assumptions relieved somewhat, by reducing, through an iterative process, the major period t_m until it approaches the largest $\bar{\theta}_{Rj}$.